SOLUTION OF EXERCISES

IN

HALL & STEVEN'S GEOMETRY,

PART III

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SOLUTIONS OF EXERCISES

IN

HALL AND STEVEN'S GEOMETRY

PART III.

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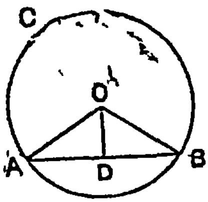
1. Draw a line A B=8 cm. and bisect it at D. At D draw D O perp. to A B making D O =3 cm. Join O A and O B.



Now, the $\triangle^s$ O A D and O D B are identically equal (Theor 4). $\therefore$ O A=O B.

With centre O and radius O A or O B draw the circle A B C. Then A B C is the required circle.

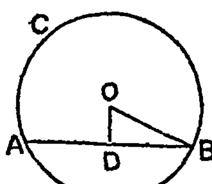
It is required to find the length of O B and verify it by measurement.

From Theor. 29 we have,

$O\ B=\sqrt{DB^2+OD^2}=\sqrt{4^2+3^2}=\sqrt{25}=5$cm.

Measure O B and it will be found to be 5 cm. long.

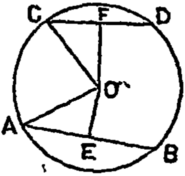
2. Take any point O, and with O as centre and radius = 13″ describe a circle ABC. From O draw a st. line O D=5″. Through D draw a st. line A D B perp. to O D, meeting the circumference at A and B. Then A B is the required chord. Join A B.



Then from Theor. 29, $D\ B=\sqrt{OB^2-OD^2}$
$=\sqrt{13^2-5^2}=\sqrt{144}=12''$.

Now, A B=2 DB (Converse, Theor. 31), $=2\times 12$ or $24''$.

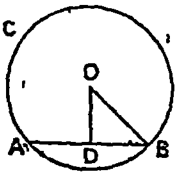
3. Take any point O, and with O as centre and radius $=1''$ describe a circle A B D C. Take any two points A and C on the circumference. With A and C as centres and radii $=1·6''$ and $1·2''$ respectively draw arcs cutting the circle at B and D. Join AB and CD. Then A B, C D are the required chords. From O draw O E perp. to A B, and O F perp. to C D. Join O A and O C.



Then from Theor. 29, we have $OE=\sqrt{OA^2-AE^2}$ $=\sqrt{1^2-·8^2}=\sqrt{·36}=·6''$, and $OF=\sqrt{OC^2-CF^2}$ $=\sqrt{1^2-·6^2}=\sqrt{·64}=·8''$.

Measure O E and O F and it will be found that O E $=·6''$, and O F $=·8''$

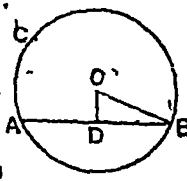
4. Take any pt O, and with O as centre and radius $=4$ cm. draw a circle A B C. Take any point A on the circumference. With A as centre and radius $=6$ cm. draw an arc cutting the circle at B. Join A B. Then A B is the required chord. From O draw O D perp. to A B. Join A B.



From Theor. 29, we have $OD=\sqrt{OB^2-DB^2}$ $=\sqrt{4^2-3^2}=\sqrt{7}=2·6$ cm. approx

Measure O D and it will be found to be 2·6 cm. nearly.

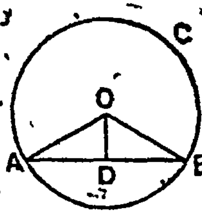
5. With any pt. O as centre and radius $=3·7$ cm. draw a circle A B C. With any pt. A on the circumference as centre and radius $=7$ cm. draw an arc cutting the circle at B Join A B Then AB is the required chord. From O draw OD perp to AB.



From Theor. 29, we have $OD=\sqrt{OB^2-DB^2}=$ $\sqrt{3·7^2-3·5^2}=\sqrt{1·44}=1·2$ cm.

Measure O D and it will be found to be 1·2 cm.
$\therefore$ the true length of O D $= 12''$ or 1 ft.

6. With any pt. O as centre and radius $= 1{\cdot}3''$ describe a circle A B C. With any pt. A on the circumference and radius $= 2{\cdot}4''$ draw an arc cutting the circle at B. Join A B. Then AB is the chord. Join O A, A B. It is required to find the area of the $\triangle$ AOB in sq. in.

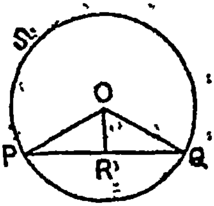


From O draw O D perp. to A B.

From Theor. 29, we have $OD = \sqrt{OB^2 - DB^2}$ $= \sqrt{1{\cdot}3^2 - 1{\cdot}2^2} = \sqrt{{\cdot}25} = {\cdot}5''$.

Area of the $\triangle$ A O B $= \frac{1}{2}$ A B . O D $= \frac{1}{2} \times 2{\cdot}4 \times {\cdot}5$ $= 6$ sq. in.

7. Let P and Q be two pts. $3''$ apart. Join P Q and bisect it at R. At R draw R O perp. to P Q. With P as centre and radius $= 1{\cdot}7''$ draw an arc cutting R O at O. Join O P and O Q. With centre O and radius $= 1{\cdot}7''$ draw a circle. This circle will pass through the points P and Q; because, the $\triangle$s O P R and O Q R being identically equal (Theor. 4), O P = O Q.



From Theor. 29, we have

$OR = \sqrt{OQ^2 - RQ^2} = \sqrt{1{\cdot}7^2 - 1{\cdot}5^2} = \sqrt{{\cdot}64} = {\cdot}8''$.
Measure O R and it will be found to be $\cdot 8''$.

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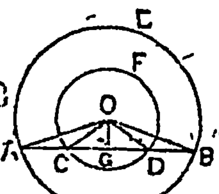
1. Let ABE and CDF be two concentric circles with O as their common centre. Let ACDB be a st. line cutting the two circles at A, B, C and D.



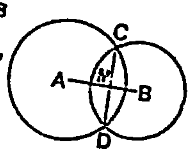
It is required to prove that the intercepts A C and D B are equal. From O draw O G perp. to A B

Proof—Then A G=G B and C G=G D (Theor. 31).

∴ A G—C G=G B—GD or A C=D B

Q E D.

2. Let two circles whose centres are at A and B intersect at C, D. Join C D and bisect it at M Join A M and B M,



It is required to prove that A M and B M are in the same st. line.

Proof—Because the st. line A M drawn from the centre A bisects the chord C D

∴ the ∠A M D is a rt. ∠ (Theor 31).

Similarly, the ∠ B M D is a rt. ∠ (Theor 31)

∴ the ∠s A M D and B M D together=2 rt. ∠s.

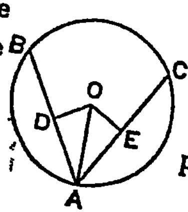
∴ A M and B M are in the same st line. (Theor 2)

Hence, *it is required to prove that the line of centres bisects the common chord at rt. angles.*

Because A B (which is the line of centres) is perp. to C D, and passes through M, the middle point of C D, (pr oved above), it bisects the common chord C D at right angles.

Q. E D.

3. Let AB, AC be any two equal chords, of circle A B C whose centre is O. It is required to show that the st line which bisects the ∠ B A C passes through the centre O.



From O draw O D perp. to A B and O E perp. to AC, Join A O.

Proof—Since O D, O E are perps. to A B, A C respectively,

∴ A B is bisected at D and A C at E. (Theor. 31, Converse). But A B = A C (given),

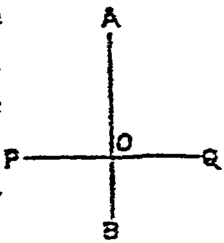
∴ their halves A D, A E are also equal.

Now, in the $\triangle^s$ O D A, O E A,

because $\begin{cases} \text{D A} = \text{A E (proved)} \\ \text{A O is common to both, and} \\ \text{the } \angle \text{O D A} = \text{the } \angle \text{O E A, being rt. } \angle^s \end{cases}$

∴ two $\triangle^s$ are equal in all respects (Theor. 18), so that the $\angle$D A O = the $\angle$ O A E. Hence A O bisects the $\angle$B A C. ∴ the bisector of the $\angle$B A C passes through the centre O Q. E. D.

4. Let P, Q be any two given points. It is required to find the locus of the centres of all circles which pass through P and Q. Join PQ and bisect it at O. Through O draw A O B perp. to P Q.



Proof—Since A O B bisects P Q at right $\angle^s$,

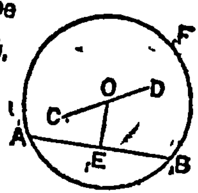
then A O B is the locus of all points equidistant from P and Q (Prob. 14).

Now, the centre of every circle passing through P and Q is a point equidistant from P and Q.

∴ the locus of the centres of all circles passing through the points P and Q is the st. line A O B which bisects P Q at right angles.

Q. E. D.

5. Let A and B be any two given points and C D a given st. line.



It is required to describe a circle passing through two points A and B and having its centre on the st. line C D.

Construction.—Join A B and bisect it at E. At E draw E O perp. to AB meeting C D at O.

Since E O bisects A B at rt. $\angle$s at E.

The centre of the circle passing through A and B, lies on the st. line O E (proved in exercise 4).

The centre also lies on the given st. line C D (Hypothesis).

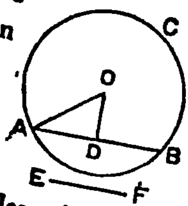
∴ the pt. O, common to both E O and C D, is the required centre.

Now with centre O and radius O A or O B describe the required circle A B F.

Q. E. F.

This problem is impossible when the given st. line C D does not meet E O, i. e., is parallel to E O, i. e., is perp. to the line A B and does not pass through the mid. pt of A B.

6. Let A and B be any two given points and E F a given st. line.



It is required to describe a circle passing through the points A and B and having a radius = E F.

Construction.—Join A B and bisect it at D. At D draw D O perp to A B. With centre B and radius = E F draw an arc cutting D O at O.

Since O D bisects A B at rt. $\angle^s$,

∴ the centre of the required circle passing through A and B lies on the st. line D O (proved in exercise 4). And since O A is equal to the st. line E F (by construction),

∴ O is the centre of the required circle.

Now, with centre O and radius O A describe the required circle A B C.

Q. E. F.

This problem is *impossible* when the given st. line EF is less than A D, *i. e*, less than half the st. line A B; for then the arc drawn with B as centre would not cut D O, and the construction would fail.

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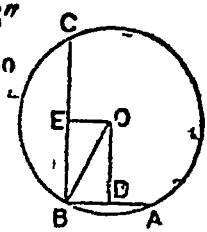
1. Let A B=1·6″ and BC=3″ be two st. lines at rt. $\angle^s$ to each other.



It is required to draw a circle passing through the points A, B and C, to find the length of the radius of the circle and to verify it by measurement.

The locus of centres of the circles passing through the points C and B is the st. line E O which bisects CB at rt. $\angle^s$ at E. Similarly, the locus of centres of the circles passing through the points B and A is the st. line D O bisecting B A at rt. $\angle^s$ at D.

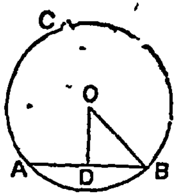
∴ the point O, common to both EO and DO, is the centre of the required circle, passing through A B and C.

Now, with centre O and radius O B draw a circle. It will pass through C and A also. Join O B.

Radius O B $= \sqrt{O\ E^2 + E\ B^2} = \sqrt{B\ D^2 + E\ B^2} = \sqrt{\cdot 8^2 + 1\cdot 5^2} = \sqrt{2\cdot 89} = 1\cdot 7.''$

Measure O B and it will be found to be 1.7″.

2. Draw a st. line A B=6 cm. and bisect it at D. At D draw D O perp. to A B, making D O=3 cm.

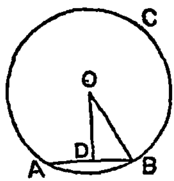


With centre O and radius= O A or O B draw the circle A B C Join O B

Radius O B$=\sqrt{OD^2+DB^2}=\sqrt{3^2+3^2}=\sqrt{18}=4{\cdot}2$ cm. nearly.

Measure O B and it will be found to be 4·2 cm.

3. With any point O as centre and radius =4 cm. draw the circle A B C. Take any point B on the circumference. With centre B and radius =4 cm. draw an arc cutting the circle at A Join A B.

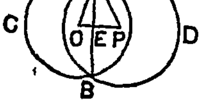


Then A B is the required chord. Join O B From O draw O D perp to A B

O D$=\sqrt{OB^2-DB^2}=\sqrt{4^2-2^2}=\sqrt{12}=3{\cdot}5$ cm. nearly.

Measure O D and it will be found to be 3 46 cm.

4. With any point O as centre and radius=2 5cm. describe the circle ABC Take any pt. A on the circumference of the circle. With centre A and radius =4 8 cm. draw an arc cutting the circle at B Join AB. From



O draw O E perp to A B, then O E will bisect A B at E (converse, Theor 31) With centre B and radius=2 6 cm. draw an arc cutting O E produced at P.

With centre P and radius=2 6 cm draw a circle. Then it will pass through the points A and B. Join A O and A P.

It is required to find the distance O P between the centres of two circles A B C and A B D and verify the result by measurement.

$O E=\sqrt{A O^2-A E^2}=\sqrt{2{\cdot}5^2-2{\cdot}4^2}=\sqrt{{\cdot}49}=7$ cm.

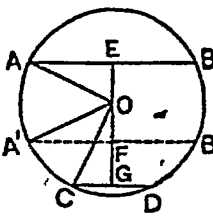
$EP=\sqrt{A P^2-A E^2}=\sqrt{2{\cdot}6^2-2{\cdot}4^2}=\sqrt{1{\cdot}00}=1{\cdot}0$ cm.

$\therefore O P=O E+E P=1{\cdot}7$ cm.

Measure O P and it will be found to be 1·7 cm.

$\therefore$ the true distance between the centres of the circles $=17''$

5. With O as centre and radius $=6{\cdot}5''$ draw the circle A C D B. Take any pt. A on the circumference. With centre A and radius $=12''$ draw an arc cutting the circle at B. Join A B. From O draw O E perp. to A B.



From E A and E B cut off lengths each $=2{\cdot}5''$. From those pts draw perps. to AB cutting the circle at C and D. Join C D. Then C D is parallel to A B and is equal to $5''$. Produce E O to meet C D in G. Then O G is also perp. to C D (Theor 14). Join OA and OC. From O G cut off O F=O E. Through F draw A′F B′ parallel to A B. Then A′ B′=A B. Join O A.′

It is required to show that the distance between C D and A B or A′B′ is $8{\cdot}5''$ or $3{\cdot}5''$.

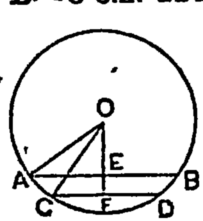
$O G=\sqrt{O C^2-C G^2}=\sqrt{6{\cdot}5^2-2{\cdot}5^2}=\sqrt{36}=6''$.

$O E=\sqrt{O A^2-A E^2}=\sqrt{6{\cdot}5^2-6^2}=\sqrt{6{\cdot}25}=2{\cdot}5''$.

$\therefore$ E G (the distance bet. A B and C D) $=E O+O G$ $=6+2{\cdot}5$ or $8{\cdot}5''$.

And F G (the distance between A′ B′ and C D) $=O G-O F=O G-O E=6-2{\cdot}5=3{\cdot}5''$.

6. Draw a st. line A B=8 cm. and bisect it at E. At E draw E F perp. to A B making E F=1 cm. Through F draw D F C parallel to A B making F C=3 cm. and F D=3 cm. Then C D=6 cm. Join A C and bisect it at right angles by a st line meeting F E



produced at O. Then O is the centre of the circle. With centre O and radius O A draw the circle A C D B.

Join O A and O C. Let O E $=x$ cm. Then OF $=$ OE $+$ EF $=(x+1)$ cm.

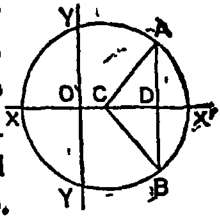
Now O A^2 $=$ O C^2 (being radii), or O E^2 $+$ A E^2 $=$ O F^2 $+$ C F^2 (Theor. 29), or $x^2+4^2=(x+1)^2+3^2$,

$\therefore$ $x=3$ cm.

$\therefore$ The radius O A $=\sqrt{x^2+4^2}=\sqrt{3^2+4^2}=5$ cm.

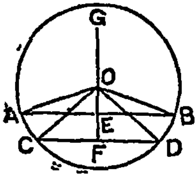
Measure O A and it will be found to be 5 cm.

7. Plot the pts. A and B whose co-ordinates are (6,5) and (6,—5) respectively. Join A B cutting X X′ at D. Then D A and D B are each $=5$. Take any point C on the x-axis. Join C A and C B. Now, the Δ^s C D A and C D B are identically equal (Theor. 4). $\therefore$ C A $=$ C B.



Hence, the circle drawn with centre C and passing through A must also pass through B.

8. Let A B, C D be any two parallel chords in the circle A C D B whose centre is O. Bisect A B at E and CD at F. Join OF and O E.



Proof—Now O E is perp. to A B, and O F is perp. to C D. (Theor. 31).

Since A B and C D are parallel, O F is also perp. to A B. Now from O two perps O E and O F are drawn to A B. Hence these lines must coincide, *i.e.*, O, E and F must be on the same st line O E F,

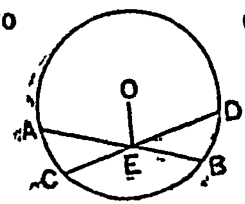
Q. E. D.

9. *See Fig. in Ex.* 8.—Let C D be any chord of the circle A C D B whose centre is O. Through O draw G O F perp. to C D cutting C D at F. Then F the mid. point of C D.

Proof.—Draw any chord A B parallel to C D, cutting F O at E. Then O E is perp. to A B. ∴ E is the mid. point of A B (Theor. 31, converse). And E lies on FG. Similarly it can be shown that the middle point of any other chord drawn parallel to C D lies on F G. Hence F G is the required locus.

Q. E. D.

10. Let A C B D be a circle whose centre is O, and let A B, C D be two chords intersecting at E.



It is required to prove that the chords A B, C D cannot bisect each other unless each is a diameter.

If possible let the chords A B, C D bisect each other at E. Join O E.

Proof—Since E is the middle point of A B, therefore ∠O E B is a rt. ∠ (Theor. 31).

Again since E is the middle point of CD, the ∠O E D is a rt. ∠ (Theor. 31).

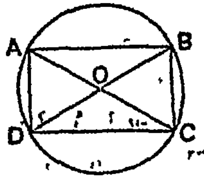
∴ The ∠ O E B = the O E D.

The part is equal to the whole, which is absurd.

Hence, A B, C D cannot bisect each other. But if each be a diameter, they would intersect at centre O. And obviously a diameter is bisected at the centre.

Q. E. D.

11. Let A B C D be a parallelogram inscribed in a circle and let the diagonals A C, B D intersect at O.



It is required to prove that O is at the centre of the circle.

Proof.—Since the diagonals AC, BD of the parallelogram bisect one another (Cor. 3, Theor. 21) at O, and each is a chord of the circle. Hence each must be a diameter (proved in Ex 10).

∴ O where the diagonals intersect is at the centre of the circle

12 *See Fig in Ex.* 11.—Let A B C D be a parallelogram inscribed in a circle and let the diagonals A C, B D intersect each other at O.

It is required to show that the parallelogram A B C D must be a rectangle.

Proof—Each of the diagonals A C, B D must be a diameter (proved in Ex 11), and hence they are equal.

∴ the parallelogram A B C D is a rectangle (Ex. 5, page 58).

Q. E. D.

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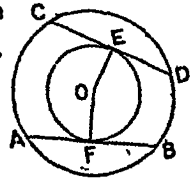
1. Let AB, CD be any two of a system of equal chords of a circle whose centre is O, and F and E be their mid. points.



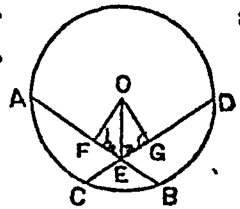
It is required to find the locus of the point E or F. Join OE and OF.

Proof—Since equal chords of a circle are equidistant from the centre, therefore OF=OE (Theor. 34).

$\therefore$ the middle point of any one of the given system of equal chords is at a distance=OF from the centre.

$\therefore$ the required locus is a circle whose centre is O and radius=OF, the common distance of the equal chords from the centre O.

2. Let AB, CD, the two chords of a circle whose centre is O, cut one another at E, such that the $\angle$AEO=the $\angle$OED.



It is required to prove that AB, CD are equal.
From O draw OF perp to AB and OG perp. to CD.
Proof.—In the $\triangle^s$ OFE and OEG

because $\begin{cases} \text{the } \angle \text{ OEF}=\text{the } \angle\text{OEG (given)} \\ \text{the } \angle \text{ OFE}=\text{the } \angle\text{OGE, being rt. } \angle^s \\ \text{and OE is common to both} \end{cases}$

$\therefore$ the two $\triangle^s$ are equal in all respects (Theor. 17); so that OF=OG.

$\therefore$ AB=CD (converse, Theor. 34).

Q. E. D.

3. *See fig. in Ex.* 2.—Let the two equal chords AB, CD of a circle whose centre is O, intersect at E.

It is required to prove that AE=ED, and EB=CE. From O draw OF perp. to AB, and OG perp., to CD. Join OE.

Proof.—Because AB=CD, $\therefore$ OF=OG (Theor. 34).
In the $\triangle^s$ OFE and OEG,

because $\begin{cases} \text{OF=OG} \\ \text{OE is common to both} \\ \text{and the } \angle\text{OFE}=\text{the } \angle\text{OGE, being rt. } \angle^s \end{cases}$

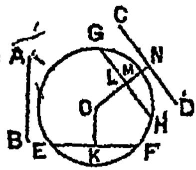
∴ the two Δ^s are congruent (Theor. 18); so that $FE = EG$.

Because OF is perp. to AB, ∴ F is middle point of AB (converse, Theor. 31).

For the same reason, G is the middle point of CD.

Now $AB = CD$ (given). ∴ $AF = FB = CG = GD$. ∴ $AF + FE = GD + EG$, *i. e.* $AE = ED$. Also $FB - FE = CG - EG$, *i. e.* $EB = CE$. Q. E. D.

4. Let O be the centre of the given circle, AB and CD two given st. lines of which AB is not greater than the diameter of the circle.



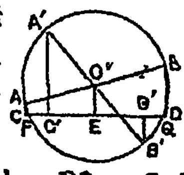
It is required to draw a chord in the given circle which shall be equal to AB, and parallel to CD.

Construction.—Take any point E on the circumference of the circle. With centre E and radius $= AB$ draw an arc cutting the circle at F. Join EF. From O draw OK perp. to EF, and ON perp. to CD. From ON cut off $OL = OK$. Through L draw HLG perp. to ON meeting the circle at G and H. Then GH is the required chord.

Proof.—Since $OL = OK$ (by construction),
∴ $GH = EF$ (converse, Theor. 34) $= AB$.
Again, since GH and CD are perps. to ON,
∴ GH and CD are parallel (Ex. 2, page 41).

Q. E. F.

5. Let PQ be a fixed chord of the circle whose centre is O; let AB and $A'B'$ be any two diameters, of which the latter cuts the chord PQ while the former does not. Draw AC, BD, $A'C'$, $B'D'$ perps. to PQ, meeting PQ produced or PQ at C, D, C', D'.



It is required to prove that the sum of the perps. AC and DB, and the difference of the perps. $A'C'$ and $B'D'$ are constant for all positions of AB.

From O draw OE perp. to PQ.

Proof—$OE = \frac{1}{2}(AC + BD)$ or $\frac{1}{2}(A'C' - B'D')$,

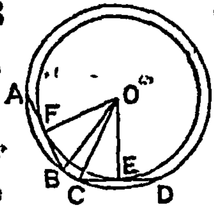
since A and B are on the *same* side, and A' and B' on *opposite* sides of PQ (Ex. 9, page 65).

Since the chord PQ is fixed (given), therefore OE its distance from the centre O is of constant length.

Hence, $(AC + BD)$ or $(A'C' - B'D')$ is constant.

Q. E. D.

6. With any point O as centre and radius $= 4 \cdot 1$ cm. draw the circle $ABCD$. With any pt. A on the circumference, as centre and radius $= 1 \cdot 8$ cm. draw an arc cutting the circle at B. Join AB. Then AB is the reqd. chord. Similarly draw the chord $CD = 1 \cdot 8$ cm. Bisect AB at F and CD at E. Join OF, OB, OC and OE.



Because OF bisects the chord AB, therefore it cuts AB at rt. $\angle^s$ (Theor. 31). And $OF = \sqrt{OB^2 - BF^2} = \sqrt{4 \cdot 1^2 - \cdot 9^2} = 4$ cm.

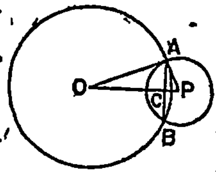
Similarly OE cuts CD at rt. $\angle^s$ (Theor 31); and $OE = \sqrt{OC^2 - CE^2} = \sqrt{4 \cdot 1^2 - \cdot 9^2} = 4$ cm. $\therefore OF = OE$.

$\therefore$ The points F, E, as well as the middle points of all chords 1·8 cm. long, lie on a circle whose centre is O and radius $= 4$ cm.

Measure OE and it will be found to be 4 cm.

With centre O and radius = ... draw the circle.

7 With any point O as centre and radius $= 3{\cdot}7''$ draw a circle. Take any pt. A on the circumference of the circle. With centre A and radius $= 2{\cdot}4''$ draw an arc cutting the circle at B. Then A and B are reqd. pts. Join AB. From O draw OC perp. to AB, then AB is bisected at C (converse, Theor. 31) Produce OC to P making $OP = 4''$. Then P is the centre of the smaller circle.



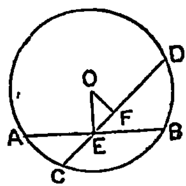
With centre P and radius $= PA$ draw a circle. This circle passes through the point B also Join OA and A P.

$OC = \sqrt{OA^2 - AC^2} = \sqrt{3{\cdot}7^2 - 1{\cdot}2^2} = 3{\cdot}5''$. $\therefore CP = OP - OC = 4 - 3{\cdot}5 = {\cdot}5''$.

$\therefore$ PA the radius of smaller circle $= \sqrt{AC^2 + CP^2} = \sqrt{1{\cdot}2^2 + {\cdot}5^2} = 1{\cdot}3''$.

Page 153.

1. Let A C B be the given circle whose centre is O, and let E be the given point in it.



It is required to draw the least possible chord through E.

Join OE. Through E draw AEB perp. to OE meeting the circle at A and B. Then AB is the reqd chord.

Let CED be any other chord through E. Draw OF perp to CD.

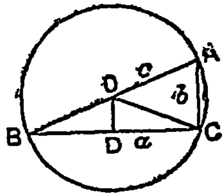
Then in the rt. angled $\triangle$ E F O, O E (being the hypotenuse) is greater than OE,

$\therefore$ CD is greater than AB (Theor. 35)

Similarly it can be proved that every other chord through the point E is greater than AB.

Hence AB is the least possible chord that can be drawn through E Q. E. F.

2. Take a st. line BC $=3\cdot5''$. With centres B and C, and radii equal to $3\cdot7''$ and $1\cdot2''$ respectively draw two arcs cutting one another at A. Join AB and AC. Then ABC is the required triangle.



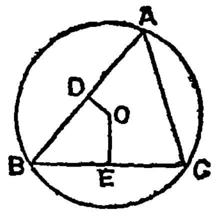
Now $a^2+b^2=3\cdot5^2+1\cdot2^2=13\cdot69=3\cdot7^2=c^2$, $\therefore$ the triangle is rt. $\angle$ d at C.

Construction—Bisect BC at D. At D draw DO perp. to BC meeting BA in O. Then O is the centre of the reqd. circle. Join OC. With centre O and radius OC draw the circle ABC which passes through A and B also, since O is the mid. pt. of AB (Ex. 10, page 47.)

$\therefore$ BA is the diameter of the circle ABC, $\therefore$ radius $=\frac{1}{2}$ BA $=\frac{1}{2}\times3\cdot7$ or $1\cdot85''$.

Measure OC and it will be found to be $1\cdot85''$.

3. Construct the $\triangle$ ABC such that AB $=3''$, BC $=2\,8''$ and AC $=2\cdot6''$. It is reqd. to draw the circum-circle of the $\triangle$ ABC and to measure its radius.

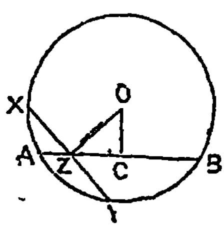


Construction—Bisect AB at D and BC at E. At D draw DO perp. to AB; at E draw EO perp. to BC meeting DO in O. Then O is the centre of the reqd. circle (Theor. 32).

With centre O and radius OA draw the circle ABC.

Measure OA and it will be found to be $1\cdot62''$.

4. Let O be the centre of the circle of which AB is the fixed chord. Take any pt. Z in AB. Join OZ. Through Z draw the chord XZY perp. to OZ



Then the chord XY has its middle pt Z (Converse, Theor 31) on AB. From O draw OC perp. to AB, then AB is bisected at C (Theor. 31).

It is reqd. to find the greatest and the least length that XY may have.

Proof.—The length of XY depends upon its distance from the centre O, *i. e.*, on OZ (Theor 31). XY will be greatest for least value of OZ, and least for greatest value of OZ.

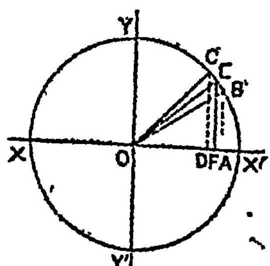
Now since Z is any pt. on AB, OZ will be least when it coincides with OC, the perp. from O to AB (Theor 12). In that case XY becomes the chord AB. Hence AB is the *greatest* length of XY.

Again, since Z must be on AB, OZ is greatest when OZ coincides with OA or OB In that case the length of the chord XY becomes zero, which is its *least* value.

Again as Z approaches from A or B to C (the foot of the perp.) length of OZ diminishes (Cor. 3, Theor. 12) ∴ XY increases as Z approaches C, the mid. pt of AB.

Q. E. D.

5. Plot the pt. B whose co-ordinates are (2·4″, 1·8″), also the pt. C whose co-ordinates are (1·8″, 2·4″).



With the origin O as centre and radius=3″ describe a circle.

Join CB and bisect it at E. From E draw EF perp. to XX′. Join OE. Draw CD, BA perps to XX′.

Because $OB = \sqrt{OA^2 + BA^2} = \sqrt{2{\cdot}4^2 + 1{\cdot}8^2} = \sqrt{9{\cdot}00} =$ 3″, and $OC = \sqrt{OD^2 + CD^2} = \sqrt{1{\cdot}8^2 + 2{\cdot}4^2} = \sqrt{9{\cdot}00} = 3''$,

$\therefore$ $OB = OC = 3''$, = the radius.

Hence, the pts. B and C are on the circle.

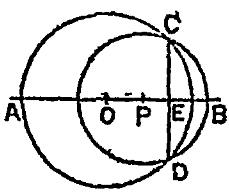
(*i*) From B draw a perp. to CD, and suppose it cuts CD at F. Then $CB = \sqrt{CF^2 + FB^2}$; but $FB = DA = OA - OD = 2 \cdot 4 - 1 \cdot 8$ or $\cdot 6''$; and $CF = CD - BA = 2 \cdot 4 - 1 \cdot 8$ or $\cdot 6''$. $\therefore$ $CB = \sqrt{\cdot 6^2 + \cdot 6^2} = \sqrt{\cdot 72} = \cdot 848'' = \cdot 85''$ approx.

(*ii*) $OF = \frac{1}{2}(OA + OD) = \frac{1}{2}(2 \cdot 4 + 1 \cdot 8) = 2 \cdot 1''$; and $EF = \frac{1}{2}(CD + BA) = \frac{1}{2}(1 \cdot 8 + 2 \cdot 4) = 2 \cdot 1''$.

(*iii*) OE (perp. from O) $= \sqrt{OF^2 + EF^2} = \sqrt{2 \cdot 1^2 + 2 \cdot 1^2} = \sqrt{8 \cdot 82} = 2 \cdot 969'' = 2 \cdot 97$ approx.

Page 155.

1. Let AB be a given st line, and C a given pt. It is reqd. to prove that all circles, whose centres lie on AB and which pass through the fixed pt. C, must pass through a second fixed point.



Draw CE perp. to AB. Produce CE to D making ED = CE. Then D is the second fixed pt.

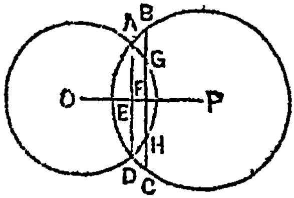
Proof—Since AB bisects CD at rt. angles,

$\therefore$ all the pts. on AB are equidistant from C and D (Prob. 14),

$\therefore$ the circles whose centres, O, P, etc., lie on AB and which pass through C, also pass through D.

Q. E. D.

2 Let two circles AGHD and ABCD whose centres are O and P intersect at A and D. Join AD. Then AD is the common chord Let a st. line BC parallel to AD cut these circles at B, G, H and C.



It is reqd. to prove that the intercepts BG and HC are equal.

Join OP cutting AD at E and BC at F.

Proof—Since OP bisects AD at rt. $\angle^s$ (Ex. 2, page 147) and BC is parallel to AD

$\therefore$ OP cuts BC at rt. $\angle^s$ (Ex, 3, page 41);

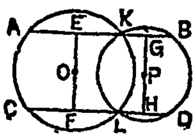
and since BC is the chord of the circle ABCD it bisects BC (Converse, Theor. 31), *i.e.*, BF = FC.

Again since GH is the chord of the circle AGHD and OF is perp. to it, OF bisects GH (Converse, Theor. 31), *i.e.*, GF = FH.

$\therefore$ BF − FG = FC − FH, or BG = HC.

Q. E. D.

3. Let two circles AKLC and KLDB whose centres are O and P cut one another at the pts. K and L. Let AKB and CLD be two parallel lines drawn through K and L, cutting the circles at A, B, C, D.



It is reqd. to prove that AB = CD.

Through O draw FOE perp to AK and CL. Through P draw HPG perp. to KB and LD.

Proof—EF and GH are parallel (Ex. 2, page 41), and AB, CD are parallel, therefore the figure EFGH is a parallelogram.

$\therefore$ EG = FH (Theor. 21)

Since OE is perp. to AK, therefore OE bisects AK at E (Converse, Theor. 31), so that EK = $\frac{1}{2}$. AK.

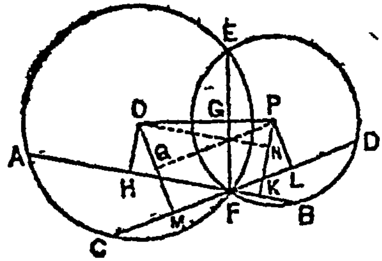
Similarly, KG = $\frac{1}{2}$. KB, FL = $\frac{1}{2}$. CL, and LH = $\frac{1}{2}$. LD.

$\therefore$ EK + KG = $\frac{1}{2}$ (AK + KB), and FL + LH = $\frac{1}{2}$ (CL + LD), *i.e.*, EG = $\frac{1}{2}$ AB, and FH = $\frac{1}{2}$ CD.

But EG = FG (proved), therefore AB = CD.

Q. E. D.

4. Let two circles ACFE and EFBD whose centres are O and P cut one another at E and F. Join EF; then EF is the common chord. Through F draw two lines AFB and CFD making equal angles with EF (*i.e.*, the $\angle$ AFE = the $\angle$ EFD and the $\angle$EFB = the $\angle$ CFE) and terminated by the circumferences at A, B, C and D.



It is reqd. to prove that AB and CD are equal.

From O draw OH, OM perps. to AF, CF respectively; from P draw PK, PL perps. to FB, FD respectively. From O draw ON perp. to PK, and from P draw PQ perp. to OM. Join OP cutting EF at G.

Proof—O P bisects EF at rt. angles (Ex. 2, page 147).

Now, in the quadrilateral OMFG, the $\angle^s$ OMF and OGF are rt angles; therefore the $\angle^s$MOG and MFG are supplementary. Similarly, in the quadrilateral GFKP, the $\angle^s$GFK and GPK are supplementary.

But the $\angle$MFG = the $\angle$GFK (given), therefore the $\angle$MOG = the $\angle$GPK.

Now, in the $\triangle^s$ OQP and ONP

because $\begin{cases} \text{the } \angle \text{ QOP = the } \angle \text{OPN (proved)} \\ \text{the } \angle \text{OQP = the } \angle \text{ ONP, being rt. } \angle^s \\ \text{and OP is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17); so that QP = ON.

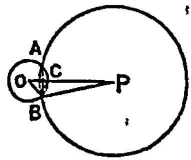
The figure OHKN is a parallelogram; $\therefore$ ON = HK (Theor. 21). Since OH is perp. to AF and PK perp. to FB, therefore OH bisects AF and PK bisects FB (Converse, Theor. 31), that is, HF is $\frac{1}{2}$. AF, and FK $\frac{1}{2}$. FB. $\therefore$ H F + F K = $\frac{1}{2}$ (AF + FB), or HK = $\frac{1}{2}$. AB. $\therefore$ ON = $\frac{1}{2}$ AB.

Similarly, it can be proved that $QP = \frac{1}{2}\,CD$.

But $QP = ON$ (proved), therefore $AB = CD$.

Q. E. D.

5. Draw a st line $AB = 2{\cdot}4$ cm Bisect AB at C Through C draw perp. OCP With centre B and radius $= 2$ cm. draw an arc cutting CO at O. With centre B and radius $= 3{\cdot}7$ cm. draw another arc cutting CP at P. With centres O and P and radii equal to 2 cm. and $3{\cdot}7$ cm. respectively draw two circles



It is reqd. to find the length of OP and verify it by measurement. Join OB and BP.

$OC = \sqrt{OB^2 - BC^2} = \sqrt{2^2 - 1{\cdot}2^2} = \sqrt{2{\cdot}56} = 1{\cdot}6$ cm.

And $CP = \sqrt{BP^2 - BC^2} = \sqrt{3{\cdot}7^2 - 1{\cdot}2^2} = \sqrt{12{\cdot}25} = 3{\cdot}5$cm.

$\therefore\ OP = OC + CP = 1{\cdot}6 + 3{\cdot}5 = 5{\cdot}1$ cm.

Measure OP and it will be found to be $5{\cdot}1$ cm.

$\therefore$ True length of $OP = 51''$.

6. *See fig. in Ex.* 5—Make a st. line $OP = 2{\cdot}1''$. With centres O and P, and radii equal to $1''$ and $1{\cdot}7''$ respectively draw two circles, intersecting at A and B. Join AB cutting OP at C. Then AB is the common chord.

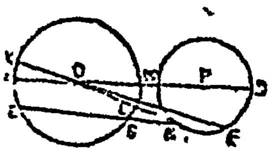
It is reqd. to find by calculation, and by measurement, the length of AB, and the lengths of OC and CP.

Join OB and BP.

Let $OC = x$, then $CP = OP - OC = 2{\cdot}1 - x$ Now, $OB^2 - OC^2 = CB^2 = BP^2 - CP^2$, or $1^2 - x^2 = 1{\cdot}7^2 - (2{\cdot}1 - x)^2$, or $4{\cdot}2x = 2{\cdot}52$, $\therefore\ x = {\cdot}6''$, *i. e.*, $OC = {\cdot}6''$. $\therefore\ CP = 2{\cdot}1 - {\cdot}6 = 1{\cdot}5''$. $\therefore\ CB = \sqrt{OB^2 - OC^2} = \sqrt{1^2 - {\cdot}6^2} = \sqrt{{\cdot}64} = {\cdot}8''$. $\therefore\ AB = 2 \times {\cdot}8 = 1{\cdot}6''$.

Page 157.

1. Let O and P be the centres of two given circles AEGB and CHFD which do not intersect. Join OP cutting the circles at B and C. Produce OP bothways to cut the circles again at A and D. Let any other st. line EGHF cut the circles at E, G, H and F. Join FO and produce it to meet the circumference at K.



It is reqd. to prove that (*i*) AD is the greatest, and (*ii*) BC the least of the st. lines which have one extremity on each of two given circles.

Proof—(*i*) Since from the external pt. O, the st. line OD is drawn through the centre P, and OF is any other line,

∴ OD is greater than OF (Theor. 37). To these unequals add equals OA and OK.

Then OD + AO are greater than OF + KO,

i.e. AD is greater than KF.

Again, since from the external pt. F, the st line FK is drawn through the centre O, and EF, is any other line,

∴ KF is greater than EF (Theor. 37).

∴ AD is much more greater than EF.

Similarly, it can be proved that AD is greater than any other line having one extremity on each of the two circles.

Hence AD is the greatest of all such lines.

(*ii*) Join HO cutting the circle AEGB at L.

Because, HL when produced passes through the centre O, and HG does not, and they are drawn from the external pt. H,

∴ HL is less than HG (Theor. 37).

Again since OC when produced passes through the centre P, and OH does not, and they are drawn from the external pt. O,

∴ OC is less than OH. And since OB=OL,

∴ OC−OB is less than OH−OL,

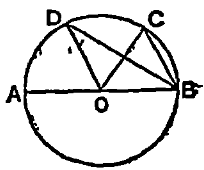
i.e., BC is less than LH; but LH is less than GH (proved). ∴ BC is much *more* less than GH.

Similarly, by taking any number of st. lines terminated by the circumferences of the circles, it can be proved that BC is less than them all.

Hence BC is the **least** of all such lines.

Q. E. D.

2. Let ABCD be a circle whose centre is O, and from any pt B on the circumference let the lines BOA, BD and BC be drawn to the circumference, so that the ∠BOD subtended by BD at the centre is greater than the ∠BOC subtended by BC.



It is reqd. to prove that of these st. lines, (*i*) BA is the greatest, and (*ii*) BD is greater than BC.

Join OD, OC.

Proof.—(*i*) In the △BOD, the sides BO, OD are together greater than BD (Theor. 11).

But OD=OA, being radii;

∴ BO, OA are together than BD,

i.e., BA is greater than BD.

Similarly it can be proved that BA is greater than any other straight line drawn from B to the circumference.

Hence, BA is the *greatest* of all such lines.

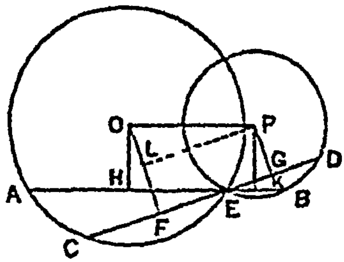
(*ii*) In the two △s DOB and COB,

because $\begin{cases} OD = OG, \text{ being radii} \\ OB \text{ is common to both} \\ \text{but the } \angle DOB \text{ is greater than the } \angle COB \end{cases}$ (given)

$\therefore$ BD is greater than BC (Theor. 19)

Q. E. D.

3. Let AEC and EDB be two given circles whose centres are O and P, and let E be one of the pts. of intersection of the circles. Join OP. Through E draw a st. line AEB parallel to OP and terminated by the circumferences at A and B.



It is reqd. to prove that AB is the greatest of all lines drawn through E.

Let CED be any other st. line drawn through E. From O draw OH, OF perps to AE, CE From P draw PK, PG, PL perps to EB, ED and OF.

Proof—Since the figures OHKP and LFGP are parallelograms,

$\therefore$ OP = HK, and LP = FG (Theor. 21)

In the rt. angled $\triangle$OLP, the hypotenuse OP is greater than LP.

$\therefore$ HK is greater than FG.

But $HK = HE + EK = \frac{1}{2} AE + \frac{1}{2} EB = \frac{1}{2} AB$;

and $FG = FE + EG = \frac{1}{2} CE + \frac{1}{2} ED = \frac{1}{2} CD$.

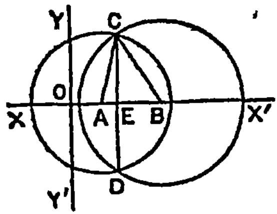
$\therefore$ AB is greater than CD,

Similarly it can be proved that AB is greater than any other st. line drawn through E and terminated by the circumferences.

Hence, AB is the greatest of all such lines.

Q. E. D.

4. Take any two pts A and B on the x-axis. Let D be the pt. whose co-ordinates are (8, −11). With centres A and B, and radii AD, BD respectively draw two circles intersecting again at the pt. C.

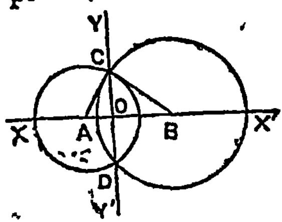


It is reqd to find the co-ordinates of C.

Join CD. Then CD is bisected at rt. angles by AB (Ex. 2, page 147), *i e.*, by the x-axis.

Hence, the co-ordinates of C are (8, 11).

5. Plot the pts. A, B and C whose co-ordinates are (−6, 0), (15, 0) and (0, 8) respectively. With centres A and B and radii AC, BC respectively draw two circles intersecting again at D.



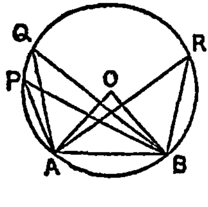
It is reqd. to find the lengths of the radii of two circles, and the co-ordinates of the pt. D.

Join AC and CB.

Because both the centres A and B lie on the axis of x, and pt. C lies on the y-axis, the st. line CD is bisected at rt. angles at the origin O by the x-axis. Therefore, the co-ordinates of the pt. D are (0, −8).

$\therefore AC = \sqrt{CO^2 + AO^2} = \sqrt{8^2 + 6^2} = 10$; and $CB = \sqrt{CO^2 + OB^2} = \sqrt{8^2 + 15^2} = 17$.

6 Let OAB be an isosceles triangle with an angle of 80° at its vertex O. With centre O and radius O A draw a circle. Let P, Q, R, . be any number of pts. on the circumference of the circle on the same side of AB as the centre O. Join AP, BP, AQ, BQ, AR, BR, ...



It is reqd. to measure the angles APB, AQB, ARB, ..., subtended by the chord AB at the pts. P, Q, R,...

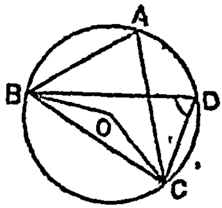
Measure the $\angle^s$ APB, AQB, ARB, and it will be found that each them is equal to 40°.

Now, make the $\angle AOB = 50°$, and repeat the same exercise. It will be found the each of the $\angle^s$ APB, AQB is equal to 25°.

Inference—The angles at circumference of a circle subtended by any chord are all equal to one another; and each of them is half of the angle at the centre subtended by the chord.

Page 161.

1. Let BAC, BDC be angles in the same segment (major) BADC of a circle whose centre is O. Join OB, OC. The angle BDC is given 74°. It is reqd. to find the number of degrees in each of the $\angle^s$ BAC, BOC, OBC.



The $\angle BAC =$ the $\angle BDC$ (Theor. 39)
$= 74°$,

the $\angle BOC = 2$ the $\angle BDC$ (Theor. 38), $= 2 \times 74 = 148°$.

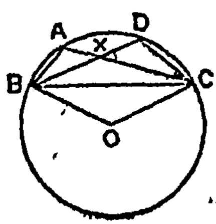
Since OB = OC, being radii, $\therefore$ the $\angle OBC =$ the $\angle OCB$ (Theor. 5).

But the $\angle^s$ BOC, OBC and OCB together $= 180°$ (Theor. 16)

$\therefore \angle OBC + \angle OCB = 180° - \angle BOC$

or, $2 \angle OBC = 180° - 148° = 32°$. $\therefore \angle OBC = 16°$.

2. Let BAC be angles in the same (minor) segment BADC of a circle whose centre is O. Join OB, OC. Let BD and CA intersect at X. The $\angle$ DXC is given 40°, and the $\angle XCD$, 25°.



It is reqd. to find the number of degrees in the $\angle$ BAC and in the reflex $\angle$ BOC.

In the $\triangle$DXC, the $\angle^s$XDC, DXC and XCD together $=180^\circ$ (Theor. 16),

$\therefore$ $\angle$XDC$=180^\circ-(40^\circ+25^\circ)=115^\circ$.

But the $\angle$ BAC$=$the $\angle$BDC (Theor. 39), $=115^\circ$.

The reflex $\angle$ BOC$=2$ the $\angle$BDC (Theor. 38),

$=2\times115^\circ=230^\circ$

3. *See fig. in Ex.* 1 —The $\angle$CBD is given 43°, and the $\angle$BCD, 82°.

It is reqd. to find the number of degrees in the $\angle^s$ ABC, OBD, OCD.

In the $\triangle$ DBC, the $\angle^s$BDC DBC and BCD together $=180^\circ$ (Theor. 16), and the $\angle$CBD$=43^\circ$, and the $\angle$BDC $=82^\circ$.

$\therefore$ the $\angle$ BDC$=180^\circ-(43^\circ+82^\circ)=55^\circ$.

$\therefore$ the $\angle$ BAC$=$the $\angle$BDC (Theor. 39), $=55^\circ$.

$\therefore$ the $\angle$BOC$=2$ the $\angle$ BDC (Theor. 38)$=2\times55=110^\circ$

Since OB$=$OC, being radii; therefore the $\angle$ OBC$=$ the $\angle$ OCB (Theor. 5). In the $\triangle$ OBC, the $\angle^s$ BOC, OBC, OCB together$=180^\circ$ (Theor 16), and the $\angle$ BOC $=110^\circ$. $\therefore$ OBC$+\angle$ OCB$=180^\circ-110^\circ$, or 2 $\angle$ OBC$=70^\circ$. $\therefore$ $\angle$OBC$=35^\circ=\angle$OCB

$\angle$ OBD$=\angle$DBC$-\angle$OBC$=43^\circ-35^\circ=8^\circ$; and $\angle$ OCD$=\angle$BCD$-\angle$OCB$=82^\circ-35^\circ=47^\circ$.

✓ 4. *See fig in Ex* 2.—It is reqd. to show that the $\angle$ OBC$=\angle$BAC-90°.

Proof—In the $\triangle$ BOC, because the $\angle^s$ BOC, OBC and OCB together$=180^\circ$, (Theor 16), and the $\angle$ OBC$=$ the $\angle$OCB (Theor 5), $\therefore$ 2 $\angle$ OBC$=180^\circ-\angle$BOC$=180^\circ-(360^\circ-$reflex $\angle$ BOC)$=$reflex $\angle$BOC-180°.

$\therefore$ $\angle$ OBC$=\frac{1}{2}$. reflex $\angle$BOC-90°.

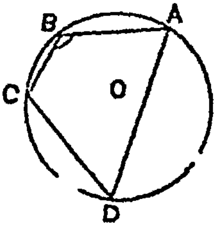
But the $\angle BAC = \frac{1}{2}$. reflex $\angle BOC$ (Theor. 38)

$\therefore$ $\angle OBC = \angle BAC - 90^\circ$.

Q. E. D.

Page 163.

1. With any pt. O as centre and radius = 1·6 draw the circle ABCD. Take two pts. B and A on the circumference. Join B A. At B make the $\angle$ ABC$=126^\circ$, the arm BC meeting the circumference at C. Take any pt. D on the arc opposite to B. Join D C and DA. Then ABCD is the reqd. inscribed quadrilateral.



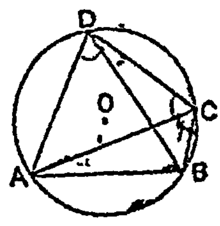
Measure the $\angle^s$ BCD, CDA, and BAD, and it will be found that the $\angle BCD = 114^\circ$, the $\angle CDA = 54^\circ$, and the $\angle BAD = 66^\circ$.

(Note—The $\angle$ ADC will always be equal to 54°; but the $\angle^s$ BCD and BAD may have different values depending on the position of D.)

The $\angle^s$ ABC and ADC$=126^\circ + 54^\circ = 180^\circ$; and the $\angle^s$BCD and BAD$=114^\circ + 66^\circ = 180^\circ$.

Hence, the opposite angles of the inscribed quadrilateral ABCD are supplementary.

2. Let ABCD be a quadrilateral inscribed in the circle ABC. Join AC, DB.



It is reqd. to prove by the aid of Theorems 39 and 18, that the $\angle^s$ ADC, ABC together$=$2rt. $\angle^s$ = the $\angle^s$ BAD, BCD together.

Proof—Since the $\angle$ADB$=$the $\angle$ACB, and the $\angle$ BDC$=$the $\angle$BAC (Theor. 39).

$\therefore$ the $\angle$ADC$=$the $\angle$ ADB$+$the $\angle$ BDC $=$ the $\angle$ ACB$+$the $\angle$BAC.

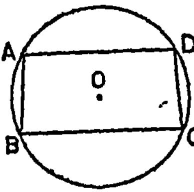
To these equals add the $\angle ABC$.

Then, the $\angle ADC +$ the $\angle ABC =$ the $\angle^s ACB + BAC + ABC = 2$ rt $\angle^s$ (Theor. 16)

Similarly it can be proved that the $\angle^s BAD$, BCD together $= 2$ rt. $\angle^s$.

Q. E. D.

3. Let ABCD be a parallelogram about which a circle can be described. It is reqd to prove that the parallelogram ABCD is a rectangle.



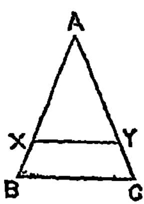
Proof—Because A B C D is a cyclic quadrilateral, therefore the opp. $\angle^s BAD$ and BCD together $= 2$ rt. $\angle^s$ (Theor 40)

But the $\angle BAD =$ the opp. $\angle$ BCD (Theor. 21).

$\therefore$ each of the $\angle^s$ BAD and BCD is a rt. $\angle$, and since quadrilateral A B C D is a parallelogram, it is a rectangle.

Q. E. D.

4. Let ABC be an isosceles triangle, and let XY be drawn parallel to the base BC, cutting the sides AB, AC in X and Y. It is reqd. to prove the four pts. B, C, X, Y lie on a circle.



Proof—Since AB = AC (given), therefore the $\angle ABC =$ the $\angle ACB$ (Theor. 5).

Since XY and BC are parallel and XB meets them,

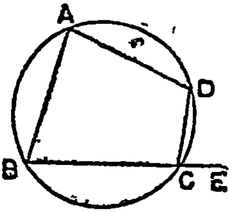
$\therefore$ the $\angle^s YXB$ and XBC together $= 2$ rt. $\angle^s$ (Theor. 14).

$\therefore$ the $\angle^s YXB$ and YCB together $= 2$ rt. $\angle^s$.

Hence, the pts. B, C, X, Y are concylic (Converse, Theor. 40).

Q. E. D.

5. Let ABCD be a cyclic quadrilateral and let BC be produced to any pt. E. It is reqd. to prove that the exterior ∠DCE = the opposite interior ∠ BAD.



Proof—Because ABCD is a cyclic quadrilateral, therefore the ∠BAD is supplement of the ∠BCD (Theor. 40).

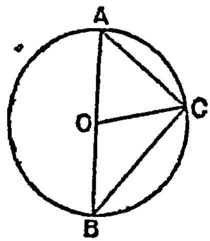
Also, the ∠ E C D is supplement of the ∠ B C D (Theor. 1)

∴ the ∠ BAD = the ∠DCE [Cor. 3, (i), Theor. 1].

Q E. D.

Page 165.

1. Let ABC be a triangle rt. angled at C. It is reqd. to prove that the circle described on the hypotenuse A B as a diameter passes through the opp. angular pt C. Bisect AB at O. Join OC.

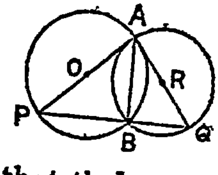


Proof—Since OC = $\frac{1}{2}$ AB (Ex. 10, page 47), therefore OC = OA = OB.

Hence, a circle described with centre O and radius OB will pass through the pts. A and C.

Q. E. D.

2. Let two circles APB and AQB, whose centres are O and R, intersect at A and B. Let two diameters AP, AQ be drawn through A.



It is reqd. to prove that the pts. P, B, Q are collinear. Join AB, PB, BQ.

Proof—Since AP is a diameter of the circle APB, therefore the ∠ABP is a rt. ∠ (Theor. 41).

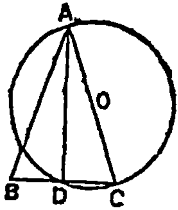
Again since AQ is a diameter of the circle ABQ, the ∠ABQ is a rt. ∠ (Theor. 41).

$\therefore$ the $\angle^s$ ABP and ABQ together $= 2$ rt. $\angle^s$

Hence, PB, BQ are in the same st. line, i.e., the pts. P, B, Q are collinear.

Q. E. D.

3. Let ABC be an isosceles triangle and on one of the equal sides AC as a diameter let the circle ACD be described cutting BC at D.



It is reqd. to prove that D is the middle pt. of B C. Join AD.

Proof—Since AC is the diameter of the circle ACD,

$\therefore$ the $\angle$ADC is a rt. $\angle$ (Theor. 41),

$\therefore$ $\angle$ ADB is also a rt. $\angle$.

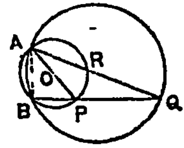
Now in the $\triangle^s$ ABD and ADC,

because $\begin{cases} \text{AB} = \text{AC (given)} \\ \text{AD is common to both} \\ \text{and the } \angle\text{ADB} = \text{the } \angle\text{ADC, being rt. } \angle^s, \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 18) so that BD $=$ DC , *i. e.* D is the mid. pt. of BC.

Q. E. D.

4. *Also see fig. in Ex.* 2.—Let APQ be a triangle. Let two circles APB and ABQ be described on AP, AQ as diameters, and let them intersect again at B.



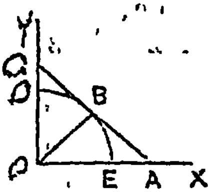
It is reqd. to prove that the point B lies on the third side PQ, or PQ produced. Join AB.

Proof—Since AP is a diameter, the $\angle$ABP is a right angle [Theor. 41]. For the same reason the $\angle$ABQ is a right angle.

And these angles have one arm AB common, ∴ the other arms BP and BQ must lie in the same str. line, since at B there can be only one perp to AB; *i.e.*, P, B and Q lie on the same str. line, *i.e.*, B lies on PQ or PQ produced.

Q. E. D.

5. Let AC denote one position of the straight rod, sliding between two straight rulers OX and OY placed at right angles to one another.



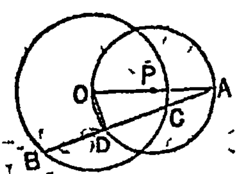
It is reqd. to find the locus of the middle point of the rod AC. Bisect CA at B. Join OB.

Proof—Then $OB=\frac{1}{2}$ AC (Ex. 10, page 47). The length of AC is constant, therefore the length of OB is also constant. And since O is a fixed point, the locus of B is a circle whose centre is O and radius $OB=\frac{1}{2}$ AC.

But since the rod AC slides between the rulers OX and OY, its middle pt. B never goes beyond these rulers. Therefore, the reqd. locus is the arc DBE.

Q. E. D.

6. Let O be the centre of the given circle and A any given pt. outside it. It is reqd. to find the locus of the middle pts. of chords of the given circle drawn through the fixed pt. A.



From A draw a st line ACB cutting the circle at C and B. Then CB is a chord through A. From O draw OD perp. to BC. Join O A.

Since OD is perp. to BC, therefore OD bisects BC at D (converse, theor. 31.)

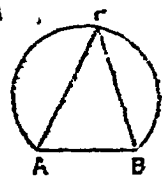
Since ODA is a rt. angled triangle, rt. angled at D,

∴ the circle ODA described upon the hypotenuse OA as diameter passes through D.

Similarly it can be proved that the middle pts. of all chords drawn through A lie on the circle ODA. And since the mid. pt. of a chord must lie within the circle the locus of the mid. pts. of all chords drawn through A, is an arc of the circle ODA described upon OA as diameter, enclosed by the given circle. The same reasoning can be applied when A is on, or within the circumference of the given circle. OA is less than, equal to or greater than the radius of the given circle, according as the pt. A lies within, on, or without the circumference of the given circle; also, in the last case when A lies without, the locus is only an arc, while in the other two cases the locus is the complete circle.

Pages 170-1

1. Let P be any pt. on the arc of a segment of which AB is the chord. join PA, PB.



It is reqd. to show that the sum of the $\angle^s$ PAB, PBA is constant.

Proof—In the $\triangle$ PAB, the sum of the $\angle^s$ APB, PAB and PBA

$=180^\circ$ (Theor. 16)

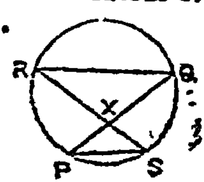
$\therefore$ the $\angle$ PAB + the $\angle$PBA $=180^\circ -$ the $\angle$ APB.

But the $\angle$ APB is constant (Theor 39)

Hence, the sum of the $\angle^s$ PAB, PBA is constant.

Q. E. D.

2 Let PQ, RS be two chords of a circle intersecting at X. Join RQ, PS.



It is reqd. to prove that the $\triangle^s$ PXS and RXQ are equiangular to one another.

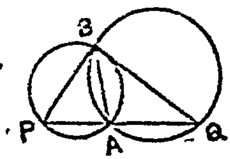
Proof.—The $\angle$ RQP = the $\angle$RSP, also, the $\angle$ QRS = the $\angle$QPS, (Theor. 39)

And, the $\angle$RXQ = the $\angle$PXS (Theor. 3)

$\therefore$ the $\triangle^s$ PXS and RXQ are equiangular to one another.

Q. E. D.

3. Let two circles intersect at A and B, and through A let any st. line PAQ by drawn terminated by the circumferences. Join PB, BQ.



It is reqd. to show that PQ subtends a constant angle at B *i.e.*, the $\angle$PBQ is constant. Join BA.

Proof—In the $\triangle$PBQ, the sum of the $\angle^s$ PBQ, BPQ and PQB = 180° (Theor. 16.)

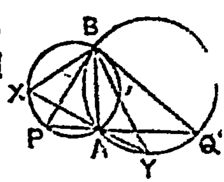
$\therefore$ the $\angle$ PBQ = 180° — ($\angle$BPQ + $\angle$PQB)

Since the chord AB is fixed, the angles in the segments APB and AQB are of constant magnitudes. (Theor. 39.)

$\therefore$ the sum of the $\angle^s$ BPA and BQA is constant, or the $\angle$ PBQ is constant.

Q. E. D.

4. Let two circles intersect at A and B, and through A let any two st. lines PAQ, XAY be drawn terminated by the circumferences.



Join PB, BQ, XB, BY.

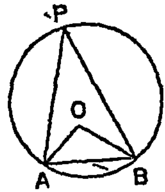
It is reqd. to show that the arcs PX, QY subtend equal angles at B, *i e.*, the $\angle$XBP = the $\angle$YBQ.

Proof—The $\angle PBX$ = the $\angle PAX$, being in the same segment PABX (Theor. 39), for the same reason the $\angle YBQ$ = the $\angle YAQ$.

But the $\angle PAX$ = the $\angle YAQ$ (Theor. 3); $\therefore$ $\angle PBX = \angle YBQ$.

Q. E. D.

5. Let P be any pt. on the arc of a segment whose chord is AB, and let the $\angle^s$ PAB, PBA, be bisected by st. lines which intersect at O. It is reqd. to find the locus of the pt. O.



Proof—In the $\triangle PAB$, $\angle APB + \angle PAB + \angle ABP = 180°$ Theor. 16)

$\therefore \frac{1}{2}\angle APB + \frac{1}{2}\angle PAB + \frac{1}{2}\angle ABP = 90°$, or, $\frac{1}{2}\angle PAB + \frac{1}{2}\angle PBA = 90° - \frac{1}{2}\angle APB$, or $\angle OAB + \angle OBA = 90° - \frac{1}{2}\angle APB$

Again, in the $\triangle OAB$, the $\angle AOB + \angle OAB + \angle ABO = 180°$ (Theor. 16.)

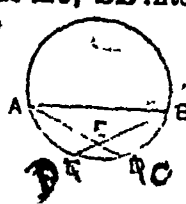
$\therefore \angle AOB + 90° - \frac{1}{2}\angle APB = 180°$, $\therefore$ $\angle AOB = 180° - (90° - \frac{1}{2}\angle APB)$ $\therefore$ $\angle AOB = 90° + \frac{1}{2}\angle APB$ = constant, (Theor. 39).

Since $\angle APB$ is constant.

Hence the locus of the pt. O is an arc of a segment on the fixed chord AB, and containing an angle $= 90° + \frac{1}{2}\angle APB$ (Converse, Theor 39).

Q. E. D.

6. Let two chords AC, DB intersect within the circle at E It is reqd. to prove that the $\angle AED$ or $\angle BEC$ = the angle at the centre, subtended by half the sum of the arcs AD and BC Join AB.



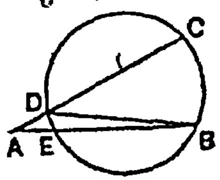
Proof—The angle at the circumference subtended by an arc = twice the angle at the circumference subtended by half the arc = the angle at the centre subtended by half the arc.

∴ the angle at the centre subtended by half the sum of the arcs AD and BC = the sum of the angles at the circumference subtended by the arcs AD and BC = the sum of the $\angle^s$ ABD and BAC = the ext. $\angle$AED (Theor. 16).

Similarly it can be proved that the $\angle$AEB = the angle at the centre subtended by half the sum of the arcs AB and DC.

Q. E. D.

7. Let two chords CD, BE intersect outside the circle at A.



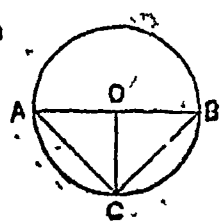
It is reqd. to prove that the $\angle$CAB = the angle at the centre subtended by half the difference of the arcs BC and DE. Join DB.

Proof—Since the angle at the centre subtended by half an arc = the angle of the circumference subtended by that whole arc (proved in Ex. 6)

∴ the angle at the centre subtended by half the difference of the arcs BC and DE = the difference of the angles at the circumference subtended by the arcs BC and DE = the difference of the $\angle^s$ BDC and DBE = the $\angle$ BAC, because the ext. $\angle$ BDC = $\angle$BAC + $\angle$DBA (Theor. 16); and ∴ $\angle$ BDC − $\angle$DBA = $\angle$BAC.

Q. E. D.

8. Let AC, CB be two chords intersecting at right angles.

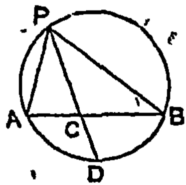


It is reqd. to prove that the sum of the arcs cut off by AC and CB = the semi-circumference.

Proof—It has been proved in Ex. 6, that the angles at the centre subtended by half the sum of the arcs by the chords = angle bet the chords = 90°, ∴ the angle at the centre subtended by the sum of the arcs = 2 × 90° = 180° ∴ the sum of the arcs = semi circumference since a semi-circumference only can subtend an angle = 180° at the centre.

Note—If the chords do not intersect the proposition does not hold. Q. E. D.

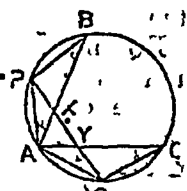
9. Let AB be a fixed chord of a circle and P any pt. on the arc APB. Join PA, PB. Let PD, the bisector of the ∠APB meet the conjugate arc ADB at D. It is reqd. to prove that for all position of P, D is a fixed pt.



Proof—Since the ∠APD = the ∠DPB (given), therefore the arc AD = the arc DB (Theor. 42).

∴ D is the mid. pt. of the arc ADB and hence it is a fixed pt. Q. E. D.

10. Let AB, AC be any two chords. Bisect the arc AB at P, and the arc AC at Q. Join PQ cutting AB at X and AC at Y. It is reqd. to prove that AX = AY. Join PB, PA, AQ, QC.



Proof—Since arc AP = arc, PB, and arc AQ = arc QC, therefore the ∠PAB = the ∠PBA, and the ∠QAC = the ∠QCA (Theor. 43).

The ∠APQ = the ∠ACQ, (Theor. 39) = the ∠QAC.

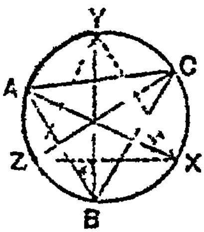
Also, the ∠PQA = ∠PBA, (There. 39), = the ∠PAB.

Now the ext. $\angle AXY = \angle^s APQ + PAB = \angle^s APQ + PQA$; also the ext $\angle AYX = \angle^s PQA + QAC = \angle PQA + APQ$ (Theor. 16).

$\therefore$ the $\angle AXY =$ the $\angle AYX$; hence $AX = AY$.

Q E. D.

11. Let ABC be a triangle inscribed in a circle. Let the bisectors of the $\angle^s$ BAC, ABC and ACB meet the circumference at X, Y and Z. Join XY, YZ and ZX.



It is reqd. to prove that, the $\angle YXZ = 90° - \frac{1}{2}$ A, the $\angle ZYX = 90° - \frac{1}{2}$ B and the $\angle YZX = 90° - \frac{1}{2}$ C.

Proof—The $\angle ZXA =$ the $\angle ZCA$, and the $\angle AXY =$ the $\angle ABY$ (Theor. 39).

The $\angle YXZ = \angle AXY + \angle ZXA$

$= \angle ABY + \angle ZCA = \frac{1}{2} B + \frac{1}{2} C.$

In the $\triangle$ ABC, the sum of the $\angle^s$ A, B, C $= 180°$ (Theor. 16)

$\therefore \frac{1}{2} A + \frac{1}{2} B + \frac{1}{2} C = 90°$

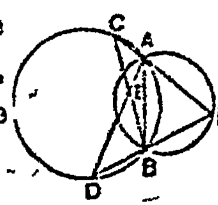
$\therefore \frac{1}{2} B + \frac{1}{2} C = 90° - \frac{1}{2} A$

$\therefore \angle YXZ = 90° - \frac{1}{2} A.$

Similarly it can be proved that the $\angle ZYX = 90° - \frac{1}{2} B$, and the $\angle YZX = 90° - \frac{1}{2} C$.

Q. E. D.

12. Let two circles ACD and ABP intersect at A and B, and let P be any pt. on the circle ABP. Join PA, PB and produce them to meet the circle ACD at C and D.



It is reqd. to prove that the arc CD is of constant length for all positions of P. Join AB, AD and CB.

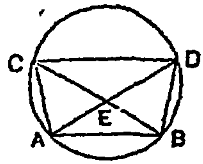
Proof.—Since the chord AB is fixed, $\therefore$ the segments A C D B and A P B are constant, and $\therefore \angle^s$ APB and ACB are constant (Theor. 39).

Now, the ext. ∠ DBC = the ∠ APB + the ∠ ACB (*Theor. 16*) = *constant*.

Hence the arc CD is constant (Converse, Theor 39).

Q. E. D.

13. Let CD and AB two parallel chords of a circle CABD. Join CA, CB, DA, DB It is reqd. to prove that CA = DB, and CB = AD.



Proof—Since the ∠DCB = the ∠CBA (Theor. 14), therefore the minor arc BD = the minor arc CA (Theor. 42),

∴ the chord DB = the chord CA (Theor. 45.)

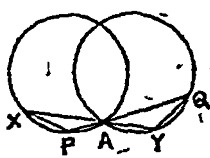
The ∠CDB is supplement of the ∠CAB (Theor. 40) and the ∠CDB is supplement of the ∠ABD (Theor. 14) ∴ the ∠CAB = the ∠ABD.

∴ the arc CB = the arc AD (Theor. 42)

∴ the chord CB = the chord AB (Theor. 45).

Q. E. D.

14. Let two equal circles XAP, AQY intersect one another at A Through A let two st. lines P A Q, X A Y be drawn terminated by the circumferences.



Join XP and YQ.

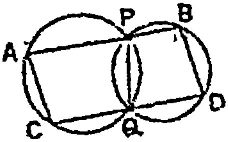
It is reqd. to prove that chord PX = chord QY.

Proof—Because the ∠XAP = the ∠QAY (Theor. 3)
∴ the arc XP = the arc QY (Theor. 42),

∴ the chord XP = the chord QY (Theor. 45)

Q. E. D.

15. Let two circles intersect at P and Q. Through P and Q let two parallel st. lines APB and CQD be drawn terminated by the circumferences. Join AC, BD.



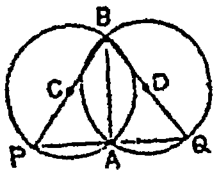
It is reqd. to prove that AC=BD. Join PQ.

Proof—Because AP is parallel to CQ, and PB is parallel to QD (given), therefore AC=PQ, and PQ=BD (Ex. 13)

∴ AC=BD.

Q. E. D.

16. Let two equal circles PBA and ABQ intersect at A and B, and through A let any st. line PAQ be drawn terminated by the circunferences. Join BP and BQ.



It is reqd. to prove that BP=BQ. Join BA.

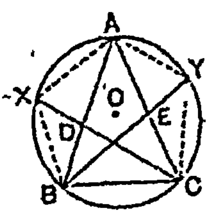
Proof—Since the circles PBA and ABQ are equal, and the chord BA is common to both,

∴ the minor arc BDA=the minor arc ACB (Theor. 44). ∴ ∠BPA=∠BQA.

∴ BP=BQ (Theor. 6).

Q. E. D.

17. Let ABC be an isosceles triangle incribed in the circles AXBCY, and let the bisectors of the base angles ACB and ABC meet the circumference at X and Y. Join AX, XB, AY, YC.



It is reqd. to prove that the four sides BX, XA, AY and YC of the figure BXAYC are equal.

Proof—The ∠ABC=the ACB (Theor. 5). ∴ their halves are equal to one another.

$\therefore$ the $\angle^s$ ABY, YBC, ACX and XCB are equal to one another.

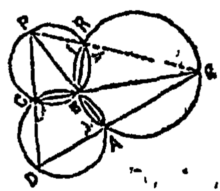
$\therefore$ the arcs on which these angles stand are also equal (Theor. 42)

$\therefore$ the chords which cut off these arcs are also equal (Theor. 45)

That is, the chords AY, YC; AX and XB are equal to one another.

Q. E D.

In order that the figure BXAYC be equilateral, the side BC must be equal to BX, $\therefore$ arc BC must = arc BX (Theor. 44) $\therefore$ $\angle$ BAC must = $\angle$BCX (Theor. 43), = $\frac{1}{2}$ $\angle$ACB = half the base angle.



18. Let ABCD be a cyclic quadrilateral; and let AB, DC be produced to meet at P, and CB, DA to meet at Q. Let the circles circumscribed about the $\triangle^s$ PBC, QAB intersect again at R. Join PR and RQ.

It is reqd. to prove that the pts. P, R, Q are collenear. Join BR.

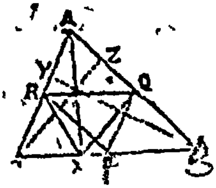
Proof—Since the quadrilateral BCPR is concyclic, $\angle$PRB is supplement to the $\angle$PCB (Theor. 40). But $\angle$DCB is supplement to the $\angle$PCB. $\therefore$ $\angle$PRB = $\angle$DCB.

Again, since the quadrilateral ABRQ is concyclic, the $\angle$BRQ is supplement to the $\angle$ BAQ. (Theor. 40). But $\angle$ BAD is supplement to the $\angle$BAQ. $\therefore$ $\angle$BRQ = $\angle$BAD.

Now the $\angle^s$ BCD + BAD = 2 rt. $\angle^s$. (Theor. 40). $\therefore$ the $\angle^s$ PRB + BRQ = 2 rt. $\angle^s$.

$\therefore$ PR and RQ are in the same st. line (Theor. 2), *i.e.*, the pts P, R, Q are collenear.

Q. E. D.



19. Let ABC be a triangle and let P, Q, R be the mid. pts. of BC, AB and AC respectively. Let X be the foot of the perp. from the vertex A on the opp. side BC. It is reqd. to prove that the four pts. P, Q, R, X are concyclic.

Join QP, QR, RP, QX and RX.

Proof—Since AXB is a rt. angled triangle, and Q is the mid. pt. of the hypotenuse AB, therefore QX=QA (Prob. 10)

∴ the ∠QXA=the ∠QAX (Theor 5).

Similarly, in the rt ∠d △ AXC, the ∠ RXA=the ∠RAX. ∴ ∠ Q X A + ∠ RXA=∠QAX+∠RAX, that is, the whole ∠ QXR = the whole ∠ Q X R = the whole ∠QAR,

Again since AQPR is a parallelogram (Ex. 2, page 64), the ∠QPR= the ∠QAR (Theor. 21)

∴ the ∠QXR=the ∠QPR

∴ the pts. P, Q, R, X are concyclic (Converse, Theor. 39)

Q. E. D.

20. *See figure in Ex.* 19.—Let ABC be a triangle and let P, Q, R be the middle pts. of BC, AB and AC respectively. Let X, Y, Z be the feet of the perps. from the vertices A, B, C on opp. sides BC, AC and AB respectively.

It is reqd. to prove that Z, Q, P, X, R, Y are concyclic. Join QP, QR, R P, QX and RX.

Proof—It has been proved in Ex. 19 that the pts. P, Q, R, X are concyclic, i. e., the circle through P, Q, R, also passes through X.

Similarly it can be proved that the circle through Q, P, R passes through Z and also through Y.

But only one circle can pass through the pts. P, Q and R. (Theor. 32),

$\therefore$ the pts. Z, Q, P, X, R, Y are concyclic.

Hence the mid. pts. of the sides of a triangle and the feet of the perps. let fall from the vertices on opp. sides are concyclic.

Q. E. D.

21. Let PAQ, PBQ be a series of triangles standing on the fixed base PQ and having their vertical $\angle^s$ PAQ, PBQ = a given angle. Let the bisectors of the vertical $\angle^s$ PAQ, PBQ meet in C. It is reqd. to prove that C is a fixed point.

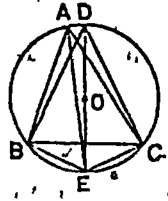
Proof—Since the base PQ is fixed and the $\angle$ PAQ = the $\angle$PBQ,

$\therefore$ the virtices A, B...of the $\triangle^s$ PAQ, PBQ,.......lie on the arc PABQ of the circle APQB of which PQ is the chord (Converse, Theor. 39.)

$\therefore$ the bisectors of the vertical angle shall in all positions of A pass through C, the mid. pt. of the minor arc, PCQ (Ex. 9, page 170).

Q. E. D.

22. Let ABC be a triangle inscribed in a circle, and let E be the mid. pt. of the arc BEC subtended by BC on the side remote from A. Through E let the diameter ED be drawn.



It is reqd. to prove that the $\angle$ DEA = $\frac{1}{2}$ ($\angle$ABC $\sim$ $\angle$ACB) Join BD, BE, DC, CE.

Proof—Because the arc BE = the arc EC (given), therefore the $\angle$BDE = the $\angle$EDC (Theor. 43).

Since DE is a diameter, therefore the $\angle^s$ DBE and DCE are rt. $\angle^s$ (Theor. 41).

Now, in the $\triangle^s$ DBE, DCE, the $\angle$ BDE = the $\angle$ EDC, and the $\angle$DBE = the $\angle$DCE (proved), therefore the $\angle$BED = the $\angle$DEC (Theor. 16, Inference 2).

The $\angle DEC = \angle AEC - \angle AED = \angle BEA + \angle AED.$

$\therefore\ 2\angle AED = \angle AEC - \angle BEA.$

But the $\angle$AEC = the $\angle$ABC and the $\angle$BEA = the $\angle$ACB (Theor 39),

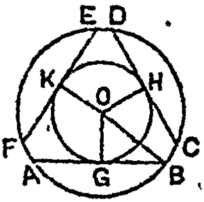
$\therefore\ 2\angle AED = \angle ABC - \angle ACB.$

$\therefore\ \angle AED = \frac{1}{2}(\angle ABC - \angle ACB).$

Q. E. D.

Page 177.

1. With any pt. O as centre and radii = 5 cm. and 3 c m. draw two concentric circles ABC and GHK. Draw series of chords of the circle ABC AB, CD, EF a touching the circle GHK at G, H, K respectively. Join OG, OH, OK, then these are perps. to AB, CD, EF respectively (Theor. 46).



Because OG, OH and OK are equal to one another being radii of the same circle $\therefore$ AB, CD and EF are equal to one another (Converse Theor. 34).

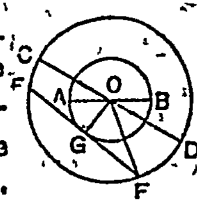
Join OB. Then GB $=\sqrt{OB^2-OG^2}=\sqrt{5^2-3^2}=4$cm. But AB = 2 GB (Converse Theor. 31) = 2×4 or 8 cm. = length of each chord of the system. On measurement each will be found to be 8 cm. long.

2. *See figure in Ex.* 1.—With any pt. O as centre and radius = 1″ draw the circle ABC. Make the chords AB, CD, EF, each = 1·6″. From O draw OG, OH, OK perps. to AB, CD, EF respectively.

Since A B = C D = E F, therefore OG = OH = OK (Theor. 34). Hence these chords touch the concentric circle GHK whose radius is OG. Join OB.

Radius OG $=\sqrt{OB^2-GB^2}=\sqrt{1^2-{\cdot}8^2}=\sqrt{{\cdot}36}={\cdot}6''$

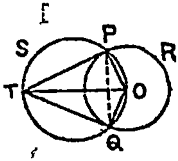
3. With any pt. O as centre and radii = 5 cm. and 2·5 cm. draw two concentric circles CED and AGB. Draw the diameters CD and AB of the two concentric circles. Draw any chord FE of the circle CED to touch the circle AGB at G. Join OG and OE.



GE $=\sqrt{OE^2-OG^2}=\sqrt{5^2-2\cdot5^2}=\sqrt{18\cdot75}=4\cdot33$ cm. nearly.

But EF $=2$ GE (Converse, Theor. 31) $=2\times4\cdot33=8\cdot7$ cm. nearly.

4. Since TSO is the circle described upon TO as a diameter, therefore the $\angle$ TPO is a rt. $\angle$ (Theor. 41)



$\therefore$ the tangent TP $=\sqrt{TO^2-OP^2}=\sqrt{13^2-5^2}=12''$.

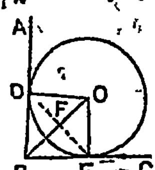
Make the st. line TO $=5\cdot2$ cm. With centre O and radius $=2$ cm. draw a circle. On TO as diameter draw the circle TSO cutting the former circle at P and Q. Join TP, TQ, PO and QO. Then TP and TQ are the two tangents.

The $\angle$ TOP = the $\angle$ TOQ (Cor. Theor. 47). Measure the $\angle$TOP and it will be found to be 67°.

5. *See figure in Ex* 4.—With any pt. O as centre and radius $=\cdot7''$ draw a circle. Take any radius OP At P draw the tangent PT $=2\cdot4''$. With centre T and radius TP draw an arc cutting the circle again at Q. Join TQ Then TQ is the other tangent. Join TO.

TO $=\sqrt{TP^2+PO^2}=\sqrt{2\cdot4^2+\cdot7^2}=\sqrt{6\cdot25}=2\cdot5''$.

6. Let AB, BC be two st. lines intersecting at A, and let O be the centre of a circle touching the lines at D and E Join B O. It is reqd. to prove that B O bisects the $\angle$ A B C. Join OD, OE.

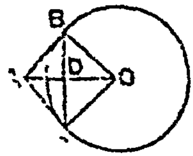


Proof—In the $\triangle^s$ DBO and OBE, because OD=OE (being radii), BO is common to both, and BD=BE (Theor. 47, Cor),

$\therefore$ two $\triangle^s$ are identically equal (Theor. 7) so that the $\angle$DBO=the $\angle$OBE.

That is, BO bisects the $\angle$ABC; or in other words, the centre O lies on the bisector of the $\angle$ ABC.

7. Let AB and AC be two tangents to a circle whose centre is O. Join BC and AO cutting one another at D. It is reqd. to prove that AO bisects the chord of contact BC at rt. angles at D. Join OB, OC.



In the $\triangle^s$ BOD and COD,

because $\begin{cases} \text{OB=OC (being radii)} \\ \text{OD is common to both} \\ \text{and the } \angle\text{BOD=the } \angle\text{DOC (Cor. Theor. 47),} \end{cases}$

$\therefore$ two $\triangle^s$ are identically equal (Theor. 4); so that the $\angle$ODB=the $\angle$ODC. The $\angle^s$ ADB and ADC being adjacent angles, each is a rt. angle.

Hence AO bisects the chord BC at rt. $\angle^s$ at D.

Q. E. D.

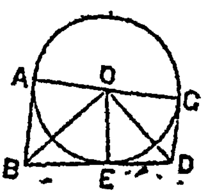
8. *See figure in Ex.* 4—Join PQ.

It is reqd. to prove that the $\angle$PTQ=2 the $\angle$OPQ.

Proof—The $\angle$OTQ=the $\angle$OPQ and $\angle$ OTP = $\angle$ OQP. (Theor. 39); also $\angle$ O P Q= $\angle$OQP (Theor 5); $\therefore$ the $\angle$ PTQ=the $\angle$OTQ+$\angle$OTP=$\angle$OPQ+$\angle$OQP =2$\angle$ OPQ.

Q. E. D.

9. Let two parallel tangents AB and CD touch the circle AEC, whose centre is O, at A and C. Let the third tangent BD touching the circle at E cut the parallel tangents AB, CD at B and D. Join OB and OD.



It is reqd. to prove that the segment BD subtends a rt angle at the centre O, *i.e.*, the $\angle$ BOD is a rt. $\angle$. Join OE.

Proof—Since BA and BE are tangents from B, $\therefore$ $\angle$AOB=$\angle$BOE (Theor. 47 Cor.), *i.e.*, $\angle$BOE=$\frac{1}{2}$ $\angle$AOE.

Similarly, it can be shown that $\angle$EOD=$\frac{1}{2}$ $\angle$EOC.

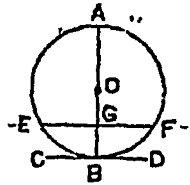
But the $\angle^s$ A O E and C O E together = 180° (Theor. 14)

$\therefore$ $\angle^s$ BOE+EOD=$\frac{1}{2}\times 180°=90°$.

i.e, $\angle$BOD is a rt angle. Hence BD subtends a rt. angle at the centre O.

Q. E. D.

10 Let AOB be the diameter of a circle whose centre is O, and let CD be the tangent to it at B.



It is reqd. to prove that the diameter AB bisects all chords parallel to the tangent CD.

Let EF be any chord parallel to CD cutting AB at G.

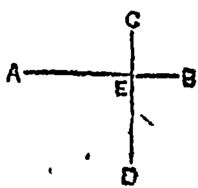
Proof—Since OB is perp to CD (Theor. 46), and EF is parallel to CD, $\therefore$ OB cuts EF at rt. angles (Ex. 3, page 41).

$\therefore$ OG bisects EF at G (Converse, Theor. 31).

Similarly it can be proved that AB bisects other chords parallel to CD.

Q E D.

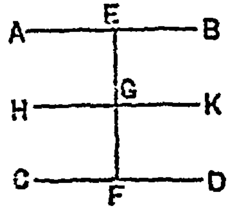
11. Let AB be a given st. line and E a given pt. in it. It is reqd. to find the locus of the centres of all circles which touch AB at the pt. E Through E draw C E D at right angles to AB Then C D is the reqd locus



Proof—Since the st. line CED is perp. to the tangent AB at the pt of contact E, it passes through the centres of the circles of which AB is a tangent at E (Cor. 2, Theor. 46).

Therefore CD is reqd. locus.

Q. E. F.



12. Let AB, CD be any two parallel st. lines. It is reqd. to find the locus of the centres of all circles touching each of the st. lines AB, CD. Take any pt. F in the st. line CD.

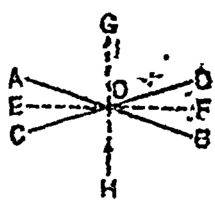
At F draw FE perp. to CD meeting AB in E. Bisect EF at G.

Through G draw HGK parallel to AB or CD. Then HK is the reqd. locus.

Proof—Since EF is perp. to CD, it is also perp. to AB (Ex. 3, page 41) Then a circle described with centre G and radius GE or GF will touch AB, CD at the pts. E, F respectively. (Theor. 46).

Thus it is evident that the centre of a circle touching two parallel st. lines is equi-distant from them; and HK is locus of each points. Hence HK is the reqd. locus.

Q. E. F.



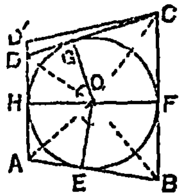
13. Let two st. lines AB, CD of unlimited length intersect at O. It is reqd. to find the locus of the centres of all circles which touch each of the two intersecting lines AB and CD.

The centre of any circle which touches two intersecting st. lines lies on the bisector of the angle between them (Ex. 6).

∴ the locus of the centres of all circles which touch each of two intersecting st. lines AB, CD is the pair of st. lines EF, GH which bisect the angles between the two given st. lines.

Q. E. D.

14. Let ABCD be a quadrilateral circumscribed about the circle EFGH, whose centre is O, and let the sides touch the circle at the pts. E, F, G and H.



It is reqd. to prove that AB+DC=DA+CB

Proof—Since from A two tangents AE, AH are drawn to the circle, EFGH, therefore AE=AH (Theor. 47, Cor.)

Similarly BE=BF, CG=CF, and DG=DH.

∴ AE+BE+CG+DG=AH+BF+CF+DH

or (AE+BE)+(CG+DG)=(AH+DH)+(BF+CF)

or AB+DC=DA+CB.

Q. E. D.

Converse—If the sum of one pair of opposite sides of a quadrilateral be equal to the sum of the other pair, then a circle can be inscribed in it.

Let ABCD be a quadrilateral in which AB+DC=DA+CB. It is reqd. to prove that a circle can be inscribed in ABCD. Bisect the $\angle^s$ D A B and ABC by st. lines AO, BO meeting at O.

Proof—Since A O, B O are the bisectors of the $\angle^s$ DAB and ABC, then O is the centre of the circle which would touch DA, AB and BC.

If this circle does not touch the side CD, let it touch the side CD′ meeting AD, or AD produced at D′.

Then AB+CD′=AD′+CB (proved)

But by hypothesis AB+DC=DA+CB

Subtracting the latter from the former, we have $CD'-DC=AD'-AD$ *i. e.*, $CD'-DC=DD'$ or $CD'=DD'+DC$ which is absurd, (Theor. 11).

Hence the circle also touches the side CD; therefore a circle can be inscribed in the quadrilateral ABCD.

Q. E. D.

15. *See figure in Ex.* 14.—Let ABCD be a quadrilateral described about the circle EFGH whose centre is O. Join OA, OB, OC and OD.

It is reqd. to prove that the $\angle^s$ DOC and AOB subtended by DC and AB at O = 2 rt. angles; also the $\angle^s$ DOA and COB subtended by AD and BC at O = 2 rt. angles. Join OE, OF, OH and OG.

Proof—Since the $\angle$ AOH = the $\angle$ AOE (Cor. Theor. 47), therefore the $\angle$AOE = $\frac{1}{2}$ the $\angle$ HOE.

Similarly, the $\angle$ BOE = $\frac{1}{2}$ the $\angle$ EOF, the $\angle$ GOC = $\frac{1}{2}$ the $\angle$FOG, and the $\angle$ DOG = $\frac{1}{2}$ the $\angle$ GOH.

$\therefore$ ($\angle$AOE + $\angle$BOE) + ($\angle$GOC + $\angle$ DOG) = $\frac{1}{2}$ ($\angle$ HOE + $\angle$EOF + $\angle$GOF + $\angle$GOH)

or $\angle$ AOB + $\angle$DOC = $\frac{1}{2}$ of 4 rt. $\angle^s$ (Cor. 2, Theor. 1), = 2 rt. $\angle^s$.

Similarly, it can be proved that the $\angle^s$ DOA + COB = 2 rt. $\angle^s$.

Q. E. D.

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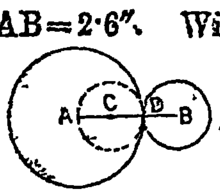
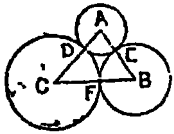
1. Take a st. line AB = 2·6″. With centres A and B, and radii = 1·7″ and ·9″ respectively draw two circles. It will be found that the circles touch externally at a point D in AB, such that AD = 1·7″ and DB = ·9″. They touch one another, because the sum of their radii = 1·7″ + ·9″ = 2·6″ = the distance between their centres [Theor. 48, Cor. (*i*)].



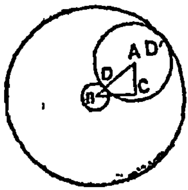
From AB cut off AC = ·8″. With centre C and radius = ·9″ draw a circle. It will be found that this

circle touches the circle, whose centre is A, internally at the pt. D This circle touches the circle with centre A, because the difference of their radii, $1\cdot7'' - \cdot8''$ = the distance between their centres [Theor. 48, Cor. (ii)].

2. Construct the $\triangle$ ABC such that BC=8 cm., AC =7 cm. and AB=6 cm. (Prob. 8). With centres A, B, and C and radii=2.5 cm., 3.5 cm., and 4.5 cm. respectively draw three circles, touching in pairs at the pts. D, E and F, because BC=8 cm.=(3.5+4.5) cm., and AC =7 cm.=(2.5+4.5) cm., and AB = 6 cm. =(2.5+3.5) cm. [Theor. 48, Cor. (i)].

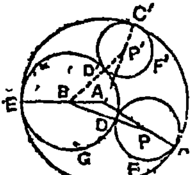


3 Take a st. line BC=8 cm. At C draw CA perp. to BC, making CA=6cm. Join AB. Then ABC is the reqd. right-angled triangle. With centre A and radius=7 cm. draw a circle, cutting AB at D.



Because $AB=\sqrt{BC^2+AC^2}=\sqrt{8^2+6^2}$ or 10 cm., if a circle be drawn with centre B to touch the former circle internal and externally, then its radius will be $10-7=3$ cm. or $10+7=17$ cm. respectively.

4. Take a st. line BA=2 cm. With centres B and A, and radii=3 cm and 5 cm. respectively draw two circles EGD′ and ECC′. Then these circles will touch each other internally at the pt E. Let P be the centre of the circle DFC which touches the circle EGD′ externally at D and the circle ECC′ internally at C Join BE, AP, BD, DP and PC. Since A and P are the centres of the circles ECC′ and DFC, and C is the pt. of contact of these two circles, therefore the pts A, P and C are in the same st. line (Theor. 48), i.e., APC is a st. line. Again since B and P are the centres of the circles E G D′ and DFC, and D is their pt. of contact therefore B D and D P are in the same st. line (Theor. 48).

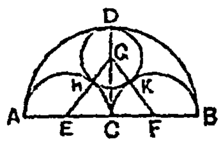


$AP=AC-PC$, $BP=BD+DP$, and $PC=DP$ being radii of same circle.

$\therefore AP+BP=AC+BD=$ sum of the radii of the given circles $=$ constant, and $=5+3=8$ cm. in this case.

Similarly if P' be the centre of any other such circle, it can be proved that $AP'+BP'=8$ cm.

5. Draw a st. line $AB=4''$ and bisect it at C. Bisect AC at E and CB at F. With centres C, E and F and radii $=2''$, $1''$ and $1''$ respectively describe the semi-circles ADB, AHC and CKB. Let G be the centre of the circle DHK touching the semi-circle ADB internally at D and the semi-circles AHC and CKB externally at the pts. H and K. Join DG, GC, GH, HE, GK and KF.



Since G and C are the centres of the circle DHK and the semi-circle ADB, and D is their pt. of contact, therefore the pts. D, G, C are in the same st. line (Theor 48). *i. e.*, DGC is a st. line. Similarly GH and HE, as well as GK and KF, are in the same st. line (Theor. 48). Since $AC=CB$, $AE=\frac{1}{2}AC$, and $CF=\frac{1}{2}CB$ therefore $EC=CF$, and hence $EH=KF$. $\therefore GH+HE=GK+KF$, or, $GE=GF$.

$\therefore$ the $\triangle^s$ GEC and GCF are congruent (Theor. 7), so that the $\angle GCE=$ the $\angle GCF$; and these being adjacent angles each is a right angle.

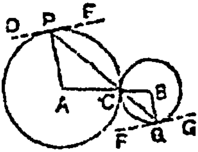
Let x'' be the length of the radius of the circle DHK, then $GC=DC-DG=(2-x)''$, and $GF=GK+KF=(x+1)''$.

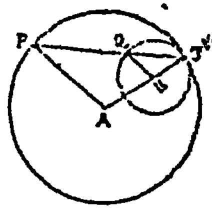
Now, $GF^2=GC^2+CF^2$, or $(x+1)^2=(2-x)^2+1^2$,

or, $x^2+2x+1=4-4x+x^2+1$

or, $6x=4$, $\therefore x=\frac{2}{3}$, $\therefore GD=\frac{2}{3}''$.

6 Let a st. line PCQ be drawn through C the pt of contact of two circles whose centres are A and B, cutting the circumferences at P and Q respectively. Join AP and BQ.





It is reqd to prove that AP and BQ are parallel. Join AC and CB

Proof—AC and CB are in the same st. line (Theor. 48).

Since AP=AC, and BC=BQ; there fore the ∠ APC= the ∠ ACP, and the ∠ BCQ=the ∠ BQC (Theor. 5)

In the case when the two circles touch each other *externally*, the ∠ ACP=the ∠ BCQ (Theor. 3). Therefore the ∠ APC=the ∠ BQC, and these being alternate angles, AP and BQ are parallel (Theor. 13).

In the case when the two circles touch each other *internally*, the ∠ACP=the ∠BCQ, being the same angle. Therefore the int. ∠ APC=the ext. ∠ BQC. Hence AP and BQ are parallel (Theor. 13).

Q. E. D.

7. *See figure in Ex 6.*—Let two circles whose centres are A and B touch externally at the pt. C, and through C, the point of contact let a st line PCQ be drawn terminated by the circumferences Let DPE and FQG be tangents to the circles at the pts P and Q respectively.

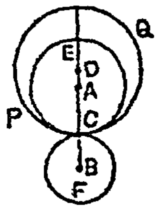
It is reqd. to prove that DE and FG are parallel. Join PA, AC, CB and BR.

Proof—Since AP and BQ are parallel (proved in Ex. 6), therefore the ∠ APQ=the ∠ PQB (Theor. 14) But the ∠s APE and BQF are equal, being rt. ∠s (Theor. 146).

∴ the remaining ∠ QPE=the remaining ∠ PQF, and these being alternate angles, DE and FG are parallel (Theor. 13)

Q. E. D.

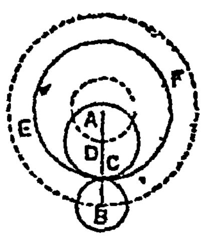
8. (i) Let D be the centre of the given circle PCQ, and C a given pt. on it. It is reqd. to find the locus of the centres of all circles which touch the given circle PCQ at C.



Let A be the centre of a circle touching the circle PCQ at C. Join DC, AC. Then D, A and C are in one st. line (Theor. 48).

i e. A lies on CD or CD produced bothways, and since C and A are given pts. the line EDCF is fixed. ∴ A always lies on a fixed line EF, which is, therefore the reqd. locus. Q. E. D.

(ii) Let A be the centre of the given circle ECF, and let its radius be a. It is reqd. to find the locus of the centres of all circles of a given radius (suppose b) and touching the given circle ECF, internally or externally. Let D and B be the centres of circles with radius b touching the given circle ECF internally and externally at any pt C Join AC, DC and BC.

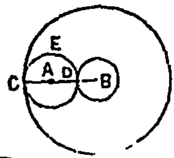


Since the circles with centres D and B touch the circle ECF internally and externally at C, therefore AC and DC, as well as AC and BC, are in one st. line (Theor. 48). Therefore AC, DC, and BC are in the same st. line.

Then $AD = AC - DC = a - b$, and $AB = AC + CB = a + b$. Now since a and b are constants therefore AD and AB are also constants; ∴ the distances of D and B from the fixed pt. A are always constants.

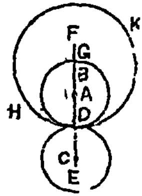
Hence the reqd. locus consists of the circles whose common centre is A, and radii equal to $(a-b)$ and $(a+b)$, as shown by dotted circles in the diagram.

9. Let A be the centre of the given circle ECD and B a given point. It is reqd. to describe a circle with centre B to touch the given circle ECD. Through A and B draw a line cutting the circle at D and C, with centre B and radii BD and BC draw two circles, then these circles will touch the given circle ECD *externally* or *internally* (Theor. 48) as the case may be. Thus there will be *two* solutions of this problem.



Q. E. F.

10 Let B be the centre of the given circle HDK of radius b, and D a given point on it. It is reqd. to describe a circle of radius a to touch the given circle HDK at D. Join BD and produce it to any pt C so that $DC = a$. From DB cut off $DA = a$. With centres C and A and radii DC and DA respectively draw two circles. These circles will touch the given circle HDK externally and internally at the pt D (Theor. 48). Thus there will be *two* solutions of this problem



Q. E. F.

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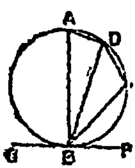
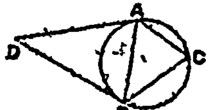
1. If the ∠ FBD = 72°, then the ∠BAD = the ∠ FBD (Theor. 49) = 72°. But the ∠s BAD and BCD together = 180° (Theor.40). Therefore, the ∠ BCD = 180°—72° = 108°. The ∠ EBD = the ∠ BCD (Theor. 49) = 108°



2. Let DA, DB be two tangents to the circle ABC from an external pt. D.



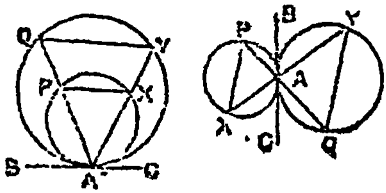
It is reqd. to prove that DA = DB. Take any pt. C on the circle ABC on the side of AB opposite to D. Join AC, BC.

Proof—The ∠ DAB = the ∠ ACB in the alt. segment, also, the ∠ DBA = the ∠ ACB (Theor. 49).

∴ the ∠ DAB = the ∠ DBA, and hence DA=DB (Theor. 6).

Q. E. D.

3. Through A the pt. of contact of two circles AQY and APX let any two chords APQ and AXY be drawn terminated by circumferences. Join BX and QY.



It is reqd. to prove that PX and QY are parallel. Draw BAC the common tangent to two circles at A.

Proof— (i) For internal contact.

The ∠ BAP=the ∠ PXA in the alt. segment of the circle APX, and the ∠ BAQ=the ∠ QYA in the alt. segment of the circle AQY (Theor. 49).

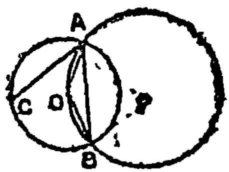
∴ the ext. ∠ PXA=the int. ∠ QYA, and hence PX and QY are parallel (Theor. 13).

(ii) For external contact.

The ∠ BAP=the ∠ PXA in the alt. segment of the circle APX, and the ∠ CAQ=the QYA in the alt. segment of the circle AQY (Theor 49). But the ∠BAP =the ∠CAQ (Theor. 8), therefore the ∠ PXA=the ∠ QYA and these being alternate angles, PX and QY are parallel (Theor. 13).

Q. E. D.

4. Let A and B be the pts. of intersection of two circles one of which passes through O, the centre of the other; let CA be the tangent to the first circle (with centre P) at A. Join AB and OA.

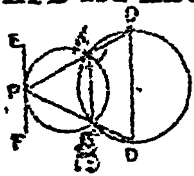


It is reqd. to prove that OA bisects the ∠ C A B. Join OB.

Proof—Since OA = OB being radii, therefore the ∠ OAB = the ∠ OBA (Theor. 5). But the ∠ CAO = the ∠ OBA in the alt. segment (Theor. 49). Therefore the ∠ CAO = the ∠ OAB, *i. e.*, AO bisects the ∠ CAB

Q. E. D.

5. Let two circles APB and ABC intersect at A and B, and through P any pt on the circle APB let the st. lines PAC, PBD be drawn to cut the circle ABD at C and D Through P draw EPF tangent to the circle APB. Join CD

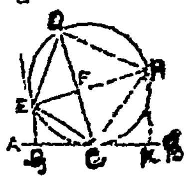


It is reqd. to prove that EF and CD are parallel. Join AB

Proof—The ∠ PAB is supplement of the ∠ BAC (Theor 1), also the ∠ BDC is supplement of the ∠ BAC (Theor. 40) ∴ the ∠ PAB = the ∠ BDC But the ∠ FPB = the ∠ PAB in the alt. segment (Theor 49). Therefore the ∠ FPB = the ∠ BDC, and these being alternate angles, EF and CD parallel (Theor. 13).

Q. E. D.

6. Let AB be a tangent to the cirle DECH at the pt. C, and from C let a chord CD be drawn. Bisect the arcs DEC and DHC at the pts. E and H respectively. From E and H draw EF, HF perp to the chord CD, and EG, HK perp to the tangent AB



It is reqd. to prove that EG = EF and HK = HF. Join EC, ED, HD and HC.

Proof—Since the arc ED = the arc EC (by construction), therefore the chord ED = the chord EC (Theor. 45). Hence the ∠ EDC = the ∠ ECD (Theor. 5) But the ∠ GCE = the ∠ EDC in the alt segment (Theor. 49). Therefore the ∠ GCE = the ∠ ECD.

Now, in the $\triangle^s$ EGC and EFC,

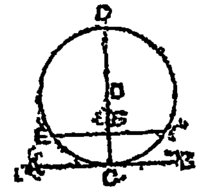
because $\begin{cases} \text{the } \angle ECG = \text{the } \angle ECF \text{ (proved)} \\ \text{the } \angle EGC = \text{the } \angle EFC, \text{ being rt. angles} \\ \text{and } EC \text{ is common to both} \end{cases}$

∴ two $\triangle^s$ are identically equal (Theor 17); so that EG = EF. Similarly it can be proved that HF = HK.

Q. E. D.

On the Method of Limits. Page 181.

2. Let DEC be a circle and DC its diameter; let ACB be drawn perp. to DC at one of its extremity C.



It is reqd. to prove that AB is tangent to the circle at the pt. C. Draw any chord EF parallel to AB cutting DC at G.

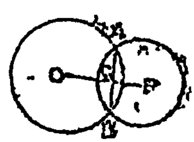
Proof—Since EF is parallel to AB, then DG is perp. to EF. Therefore EF is bisected at G (Converse, Theor. 31), and this is true *however closer G approaches to C.*

If the pt. G moves up to coincidence with C, then since EG always = GF, the pts. E and F will coincide with the pt. C, and then the chord coincides with ACB, and cut the circle at one point only.

Hence, ultimately the st. line AB is a tangent at C.

Q. E D

3. Let two circles whose centres are O and P intersect at A and B Join AB



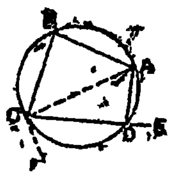
It is reqd. to prove that when the two circles touch one another the centres and the point of contact are in one st. line.

Proof—OP, the line of centres, bisects the common chord AB at right angles at C (given) *i e.*, passes through C the mid pt. of AB This is true *however near A and B approach to each other.*

If A and B come *very* close to one another and ultimately coincide, then since AC always = CB, the pt. C will also coincide with A and B, and the circles will touch each other at the pt. C.

Hence, ultimately the st. line which joins the centres of two circles touching each other, passes through the pt. of contact.

Q. E. D.



4. Let ABCD be a cylic quadrilateral, and let the side CD be produced to any pt. E; then the ext. ∠ ADE = the opp. int. ∠ A B C (Ex. 5, p. 163).

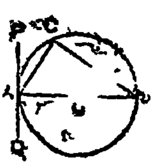
It is reqd. to deduce Theorem 49, from the above data.

Proof—The ∠ ADE = ∠ ABC (given). This is true *however near D approaches to C.*

If D moves up to and coincides with C, the chord AD will ultimately become the chord AC, the line CDE will become the tangent CE′, and the ∠ADE will become the ∠ ACE′.

Hence, ultimately the ∠ ACE′ = the ∠ ABC in the alt. segment.

Q. E. D.



5. Let CAB be a circle and AB its diameter. Take any pt. C on the circumference of the circle. Join AC and BC Then ACB is a rt. angle (Theor. 41) It is reqd. to prove that the tangent at any pt. of a circle is perp. to the radius drawn to the pt. of contact.

Proof—the $\angle ACB$ is a rt. angle (given). This is true *however near C approaches to A.*

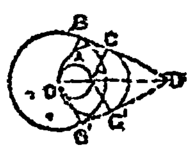
If C moves up to coincidence with A the chord BC will become the diameter BA, the chord C A will become the tangent AP, and the $\angle BCA$ will become the $\angle BAP$ $\therefore$ $\angle BAP$ is a rt. angle.

Hence the tangent PAQ at the pt. A of the circle AB C is perp. to the diameter BA (and therefore to the radius OA) drawn to the pt. of contact. Q. E. D.

Page 187.

1. There can be drawn—

(*i*) Two *direct common* tangents and no transverse, when the given circles intersect, (*ii*) Three *common* tangents—*two direct* and *a third at the point of contact*—when the circles have external contact, (*iii*) One *Common* tangent—at the point of contact—when the circles have internal contact.

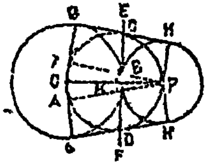


(*i*) Draw a st. line $OP = 1''$. With centres O and P, and radii $= 1{\cdot}4''$ and $1''$ respectively draw two circles. The circles intersect one another at two points.

Upon OP as diameter describe the circle AOA′P. With centre O and radius = the difference of two given radii $(1{\cdot}4''-1'') = {\cdot}4''$ draw arcs cutting the circle AOA′P at the pts. A and A′. Join OA and OA′ and produce them to meet the circumference of the larger circle at B and B′. From P draw the radii PC parallel to OB and PC′ parallel to OB′. Join BC and B′C′. Then BC and B′C′ are the two *direct common* tangents.

There will be no transverse common tangents, for P will lie within the circle of construction for transverse angents.

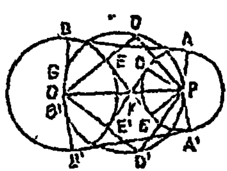
(ii) Draw a st. line OP=2·4". With centres O and P, and radii=1·4" and 1" respectively draw two circles. GG'B and H B H' The circles touch each other externally at the pt. B [Cor. (i) Theor. 48]



Upon OP as diameter describe the circle CODP. With centre O and radius=the difference of two given radii (1·4"−1")= 4" draw arcs cutting the circle CODP at A and A'. Join OA and OA', and produce them to meet the circle GG'B at G and G'. Draw the radii PH parallel to OG and PH' parallel to OG' Join GH, G'H'. Then GH, G'H' are the two *direct common* tangents. Draw EBF perp. to OP. Then EF is a transverse *common tangent* to the given circles at B, their pt. of contact, for P is on the circle of construction for transverse tangents.

(iii) Draw a st. line PO=4". With centres O and P, and radii=1.4" and 1" respectively draw two circles The circles touch each other internally at the pt. A [Cor. (ii) Theor. 48] Join PA. Then AP and PO are in one st. line (Theor. 48). Through A draw BAC perp. to AO. Then BC is the direct *common tangent* to the given circle at A their pt. of contact, since P is on the circle of *construction. There are no transverse common tangents.*

(iv) Draw a st. line PO=3". With centres O and P, and radii = 1·4" and 1" respectively draw two circles BE' EB and AA' C'C. The circles neither cut nor touch each other.



Upon OP as diameter describe the circle DPD'O. With centre O and radius = the difference of two given radii (1·4"—1"),=·4" draw arcs cutting the circle DPD'O

at G and G'. Join OG and OG' and produce them to meet the circle BE'EB at B and B'. Draw the radii PA parallel to OB and PA', parallel to OB'. Join AB, A'B'. Then AB and A'B' are the two direct common tangents.

With centre O and radius = the sum of two given radii $(1·4''+1'') = 2·4''$ draw arcs cutting the circle DPD'O at D and D'. Join OD and OD' cutting the circle BEE'B at E' and E respectively. Draw the radii PC parallel to OD' and PC' parallel to OD on opp. sides of OP. Join CE and C'E'. Then CE and C'E' are two transverse common tangents. In this case there are four common tangents.

2. *See figure in Ex.* 1, (i)—Draw a st. line $OP=2''$. With centres O and P and radii $=2''$ and $·8''$ respectively draw two circles. The circles intersect each other at two pts.

Draw the common tangent, as in Ex. 1, (i). In this case, OA or $OA'=(2''-·8'')=1·2''$.

$$BC=PA=\sqrt{OP^2-OA^2}=\sqrt{2^2-1·2^2}=\sqrt{2·56}=1·6''$$

Also $B'C'=1·6''$. Measure BC and B'C' and it will be found that each of them $=1·6''$.

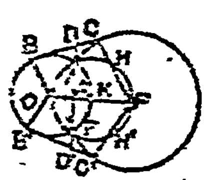
3. *See figure in Ex.* 1, (ii)—Draw a st. line $OP=1·8''$. With centres O and P, and radii $=1·2''$ and $·6''$ respectively draw two circles. The circles touch each other externally at the pt. B [Cor. (i), Theor. 48].

Draw the common tangent, as in Ex. 1, (i). In this case OA or $OA'=(1·2''-·6'')=·6''$.

$$GH=AP=\sqrt{OP^2-OA^2}=\sqrt{1·8^2-·6^2}=\sqrt{2·88}=$$

$1·7''$ nearly. Also, $G'H'=1·7''$ nearly.

Measure GH, G'H', and it will be found that each of them $=1·7''$.



4. Draw a st. line $OP=2·1''$. With centres P and O, and radii $= 1·7''$ and $1''$ respectively draw two circles EBB'F and CEFC', cutting one another at E and F.

Upon OP as diameter describe a circle. With centre P and radius = the difference of the given radii ($1.7'' - 1''$) $= .7''$ draw arcs cutting the circle (with diameter OP) at H and H′. Join PH, PH′ and produce them to meet the circle CEFC′ at C and C′. Draw the radii OB parallel to PC and OB′ parallel to PC′.

Join BC, B′C′. Then BC, B′C′ are the two *direct* common tangents.

$BC = O'H = \sqrt{OP^2 - HP^2} = \sqrt{2.1^2 - .7^2} = \sqrt{3.92} =$ $1.98''$ nearly. Also $B'C' = 1.98''$ nearly. Measure BC, B′C′, and it will be found that each of them $= 1.98''$.

Join EF cutting OP at A. Let $OA = x$, then $AP = OP - OA = 2.1 - x$. But $OE^2 - OA^2 = AE^2 = EP^2 - AP^2$, or $1^2 - x^2 = 1.7^2 - (2.1 - x)^2$, or $3.2\,x = 2.52$,

$\therefore\ x = .8''$.

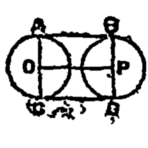
But $EF = 2\,OA = 2 \times .8 = 1.6''$. Measure EF and it will be found $= 1.6''$.

Produce EF bothways to meet BC, B′C′ at D and D′ respectively. Measure BD, DC, B′D′ and D′C′, and it will be found that $BD = DC$ and $B'D' = D'C'$. Hence DD′ bisects the common tangents BC, B′C′.

5. *See figure in Ex 1, (iv)*—Draw a st. line $OP = 3''$. With centres O and P, and radii $= 1.6''$ and $.8''$ draw two circles. The circles neither cut nor touch each other.

Draw all the common tangents, as in Ex. 1, (iv). In this case OG or $OG' = (1.6'' - .8'') = .8''$ and OD or $OD' = (1.6'' + .8'') = 2.4''$.

6. Take a st. line OP of any length. With centres O and P, and radii of equal length draw two equal circles. Through O and P draw AOC, BPD diameters of these circles, each perp. to OP. Join AB, CD. Then AB, CD are the two reqd. *direct common* tangents.



7. *See figure in Ex.* 1, (i). It is reqd. to prove that the two direct common tangents BC, B′ C′ are equal. Join AP, A′P.

Proof—Since OA = OA′, OP is common, and $\angle^s$ OAP and OA′P are rt. angles, the two $\triangle^s$ OAP and OA′P are equal (Theor. 18): so that AP = A′P. But AP = BC, and A′P = B′C′ $\therefore$ BC = B′C′.

See figure in Ex. 1, *iv*)—It is reqd. to prove that the two transverse common tangents CE and C′E′ are equal. Join PD, PD′.

Proof—Since the $\angle$ PDO is a rt. angle (Theor. 41) and the $\angle$ C′E′O is a rt. angle (Theor 46), therefore the $\angle$ C′E′O = the $\angle$ PDO. Hence PD, C′E′ are parallel (Theor 13). But PC′ is parallel to OD (by construction); therefore the figure DPC′E′, is a parallelogram. Therefore PD = C′E′ (Theor. 21). Similarly, PD′ = CE.

But $PD = \sqrt{PO^2 - DO^2}$, $PD' = \sqrt{PO^2 - D'O^2}$ and DO = D′O (by construction). $\therefore$ PD = PD′, $\therefore$ C′E′ = CE.

Q. E. D.

8. *See figure in Ex* 1, (i). Produce BC, B′C′ to meet at D. Join OD, PD. It is required to prove that OD and PD are in the same st. line.

Proof—The $\triangle^s$ BOD and B′OD are identically equal (Theor. 18), because OB = OB′, OD is common to both and the $\angle$ OBD = the $\angle$ OB′D being rt. angles (Theor. 46), so that the $\angle$BDO = the $\angle$ B′DO That is, OD bisects the $\angle$BDB′.

Similarly it can be proved that PD bisects the same angle. Therefore OD and PD are in the same st. line.

See figure in Ex. 1, (*iv*)—Let CE, C′E′ intersect at K. Join PK and KO. It is reqd. to prove that PK and KO are in the same st. line.

Proof—The $\triangle^s$ PCK and PC′K are identically equal (Theor. 18), because PC = PC′, PK is common to both, and the $\angle$ PCK = the $\angle$ PC′K being rt. angles (Theor. 46); so that the $\angle$ P K C = the $\angle$PKC′ = $\frac{1}{2}$ $\angle$CKC′. Similarly it can be proved that the $\angle$ EKO = the $\angle$ E′KO = $\frac{1}{2}$ $\angle$ EKE′. But the $\angle$ CKC′ = the $\angle$ EKE′.

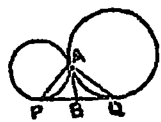
Therefore the $\angle^s$ PKC, PKC′, EKO and E′KO are all equal

Now the $\angle^s$ PKC′+PKC+CKE′=2 rt. $\angle^s$ (Theor. 1)

∴ the $\angle^s$ PKC+CKE′+E′KO=2 rt. $\angle^s$. Hence PK and KO are in the same st. line (Theor 2)

Q E D.

9. Let two given circles have external contact at A, and let PQ be a direct common tangent drawn to touch the circles at P and Q. Join AP, AQ It is reqd. to prove that the $\angle$PAQ is a rt. angle. Let the common tangent to the two circles at A meet PQ in B



Proof—Since BA and BP are two tangents from B, therefore BA=PB (Theor. 47, Cor.). Therefore the $\angle$ BAP=the $\angle$ BPA Similarly BA=BQ, therefore the $\angle$ BAQ=the $\angle$ BQA.

∴ $\angle$ BAP+$\angle$ BAQ=$\angle$ BPA+$\angle$ BQA, or $\angle$ PAQ=$\angle$ BPA+$\angle$ BQA

∴ the $\angle$ PAQ is a rt. angle (Inference 4, Theor 16)

Q. E. D.

On Loci. Foot of Page 188.

(i) *See figure in Ex.* 4, page 147.

The locus of the centres of the circles which pass through two given points is a straight line bisecting the line joining the two given points at right angles.

(ii) *See figure in Ex.* 11, page 177.

The locus of the centres of circles which touch a given straight line at a given point is a straight line perpendicular to the given straight line at the given point.

(*iii*) *See figure in Ex.* 8, (*i*), page 179.

The locus of the centres of circles which touch a given circle at a given point is the straight line passing through the centre of the given circle and the given point.

(*iv*) *See figure in Ex.* 12, page 177.

The locus of the centres of circles which touch a given straight line and have a given radius is the two straight lines parallel to the given straight line on either side of it and at a distance equal to the given radius from it.

(*v*) *See figure in Ex.* 8, (*ii*), page 179.

The locus of the centres of circles which touch a given circle and have a given radius is one or other of the two concentric circles whose radii are equal to the sum and difference of the two radii respectively.

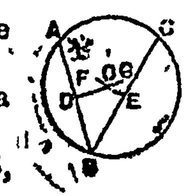
(*vi*) *See figures in Exs.* 12 and 13, page 177.

The locus of the centres of circles which touch two given straight lines is a pair of straight lines bisecting the angles between the two given straight lines.

If the given straight lines are parallel, the locus is the straight line parallel to the given straight lines and midway between them.

Page 189.

1. Let A, B, C be any three given pts. It is reqd. to draw a circle to pass through A, B, C. Join AB, BC.

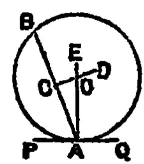


The centre of a circle passing through the pts. A, B lies on the st. line GD bisecting AB at rt. $\angle^s$ [Note (i), page 188].

The centre of a circle passing through the pts. B, C lies on the st. line FE bisecting BC at rt. $\angle^s$ [Note (i), page 188].

∴ the pt. O where the st. lines GD, FE intersect satisfies both the conditions and is therefore the reqd. centre. With centre O and radius OA draw the circle which will also pass through B and C.

2. Let A be any pt. on the st. line PQ and B any other pt. outside it. It is reqd. to draw a circle to touch PQ at A and pass through the given pt. B Join BA.

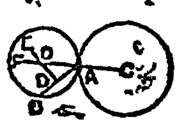


If a circle touches the st. line PQ at A, its centre lies on the st. line EA perp. to PQ at A [Note (ii), page 188.]

If a circle passes through two given pts. A and B, its centre lies on the st. line DC bisecting AB at rt. $\angle^s$ [Note (i), page 188].

∴ the pt O where the st. lines EA, DC intersect satisfies both the conditions, and is therefore the reqd. centre. With centre O and radius OA draw the circle which will touch the st. line PQ at A and pass through B.

3. Let C be the centre of the given circle and A any pt. on it. Let B be any other pt. outside the circle. It is reqd. to draw a circle to touch this circle at A and to pass through B. Join CA and produce it to any pt. F. Join AB.

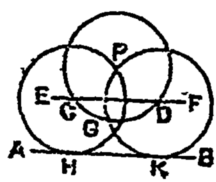


If a circle touches the given circle with centre C at the pt. A, its centre lies on the st. line through CA [Note (iii), page 188].

If a circle passes through B, A its centre lies on the st. line ED bisecting BA at rt. $\angle^s$ [Note (i), page 188].

$\therefore$ the pt. O where the st. lines CF and ED intersect satisfies both the conditions, and is therefore the reqd. centre. With centre O and radius OA draw the circle which will touch the circle with centre C at A and pass through B.

4. Let P be a pt. at a distance of 4·5 cm. from a given st. line AB. It is reqd. to draw two circles of radius 3·2 cm. to pass through P and to touch AB

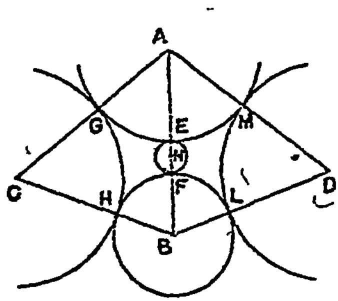


Locus of the centres of circles of radius 3·2 cm. which touch the given st line AB is a st. line EF parallel to AB situated at a distance of 3·2 cm. from it [Note (iv), page 188].

Locus of the centres of circles of radius 3 2 cm. which pass through P is a circle CGD, with centre P and radius $=3\cdot2$ cm. Let these two loci intersect at C and D.

Then C and D satisfy both the conditions, and are therefore the reqd. centres. With centres C, D and radius $=3\cdot2$ cm. draw two circles which will pass through P and touch AB at H, K.

5. Draw a st. line AB$=$6 cm. With centres A, B, and radii $=$ 3 cm. and 2 cm. respectively draw two circles. It is reqd. to draw a circle of radius 3 5 cm. to touch each of the given circles externally.



Locus of the centre of a circle of radius 3·5 cm. touching the given circle of radius 3 cm. externally is a circle whose centre is A and radius$=(3+3\cdot5)=6\cdot5$ cm. [Note (v), page 188].

Locus of the centre of a circle of radius 3·5 cm. touching the given circle of radius 2 cm. externally is a circle whose centre is B and radius$=(2+3\cdot5)=5\cdot5$ cm. [Note (*v*), page 188].

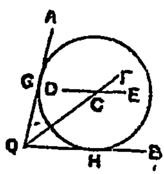
With centres A and B, and radii$=6\cdot5$ cm. and 5·5 cm. respectively draw arcs on either side of AB cutting at C and D.

Then C and D are the reqd. centres. With centres C and D, and radius$=3\cdot5$ cm. draw two circles. These circles will touch the two given circles at G, H, M, and L. Thus there are two solutions of this problem.

The centre N of the smallest circle, which touches the given circles with centres A and B externally, lies on AB midway between the pts. E and F where the given circles cut AB

$EF=AB-AE-FB=6-3-2=1$ cm. Therefore EN the radius of the smallest circle$=\frac{1}{2}$. $EF=5$ cm.

6. Make the $\angle$ AOB$=76°$. It is reqd. to describe a circle of radius 1·2″ to touch the lines OA, OB

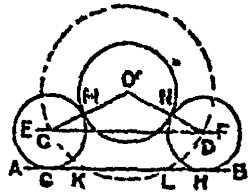


If a circle touches two st. lines OA, OB its centre lies on OF the bisector of the $\angle$ AOB [Note (*vi*), page 188].

Locus of the centre of a circle of radius 1·2″ and touching the st. line OB is a st. line DE parallel to OB at a distance of 1·2″ from it. [Note (*iv*), page 188].

$\therefore$ the pt. C where the st. lines FO, DE intersect is the reqd. centre. With centre C and radius$=1\cdot2''$ draw a circle which will touch OA, OB at G, H respectively.

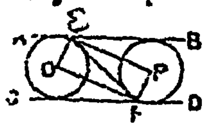
7. Let O be the centre of a given circle of radius 3·5 cm. at a distance of 5 cm. from a given st. line AB. It is reqd. to draw two circles of radius 2·5 cm. to touch the given circle and the given st. line AB.



Locus of centres of circles of radius 2·5 cm. touching the given st. line AB is a st line EF parallel to AB, at a distance of 2 5 cm. from it. [Note (*iv*), page 188].

Locus of centres of circles of radius 2 5 cm. touching the given circle with centre O is one or other of two circles whose common centre is O and radius = (3·5 + 2·5) or 6 cm. and (3 5 – 2·5) or 1 cm respectively. [Note (*v*), page 188] The first circle KCDL cuts EF at C and D; but the other does not Then C and D are the reqd. centres. With centres C, D and radius = 2 5 cm draw two circles which will touch the given circle at M, N and the given st. line AB at G, H.

8. Let AB, CD be any two parallel st. lines and EF any transversal cutting AB, CD at E, F respectively. It is reqd. to draw a circle to touch AB, CD and EF.

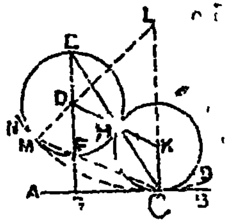


Locus of the centres of circles touching EF and CD is one or other of the st. lines FO, FP bisecting the ∠s EFC, EFD respectively [Note (*vi*), page 188].

Locus of the centres of circles touching FE and AB is one or other of the st. lines EO, EP bisecting the ∠s AEF, FEB respectively [Note (*vi*), page 188].

Hence O where FO, EO meet, and P where FP, EP meet are the reqd. centres. Since the circles touch AB and CD, their centres are equi-distant from AB and CD. Hence their radii are each = half the perp. distance between AB and CD. Draw the reqd. circles.

These circles are equal, because their radii are equal.



9. Let EMH be a given circle whose centre is D, and C a given pt. in a given st line AB It is reqd. to draw a circle to touch the given circle EMH, and the st line AB at C.

Construction—At C draw CK perp. AB then the centre of the reqd. circle lies on CK [Note (iii), page 188]. From D the centre of the circle EMH draw DG perp. to AB cutting the circle at F. Produce GD to meet the circle again at E. Join EC cutting the circle at H. Join DH and produce it to meet CK at K. Then K is the centre of the reqd. circle.

Proof—Since EG, KC are both perp. to AB, therefore they are parallel (Ex. 2, page 41)

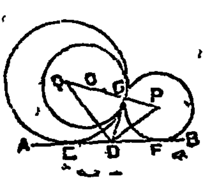
∴ the ∠ DEH = the alt. ∠ HCK (Theor. 14). Again since DE = DH, therefore the ∠ DEH = the ∠ DHE. (Theor. 5)

∴ the ∠ HCK = the ∠ DHE = the vertically opp. ∠ KHC.

∴ KH = KC. Draw a circle with centre K and radius KH Then this circle will touch the given circle EMH at H and the given st. line AB at the given pt. C

Another circle can be drawn to satisfy the given conditions. Join CF and produce it to meet the circle at M. Join DM. Produce MD, CK to meet at L. Then L is the centre of the reqd circle

Since DM = DF, the ∠ DMF = the ∠ DFM (Theor. 5) = the vertically opp ∠ GFC (Theor. 3) = the alt. ∠ MFCK Therefore LM = LC



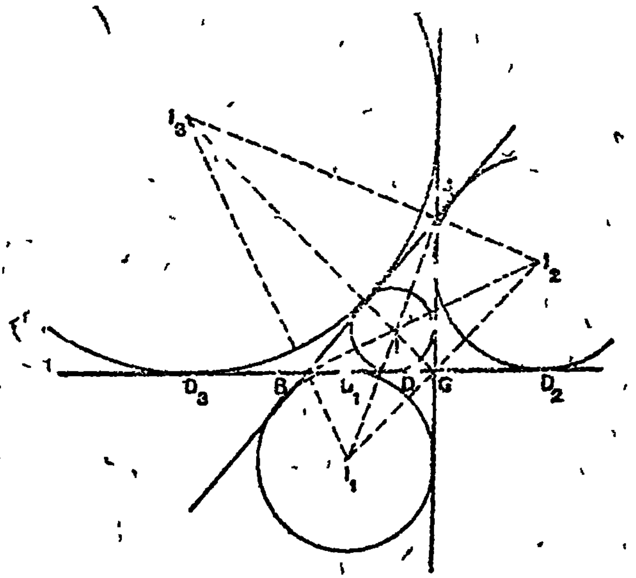
10. Let AB be a given st line and C a given pt. on a given circle whose centre is O It is reqd to draw a circle to touch AB, and also the given circle a C Join OC. At C draw CD perp. to OC,

meeting AB in D. Then CD will be the common tangent to the given circle and the reqd. one.

Centre of the reqd circle touching the given circle at C lies on the st. line through O, C. [Note (iii), page 188]

Again the centre of the reqd. circle which touches the st. lines CD, AB lies on one or other of the st. lines DP, DQ the bisectors of the ∠s CDB, CDA respectively [Note (vi), page 188]

∴ the pts. P, Q where OC produced bothways meet DP, DQ are the reqd. centres. With centres P, Q and radii PC, QC respectively, draw two circles which touch the given circle at C and the given st. line AB at F, E.



11. Let AB, BC, CA be three given st. lines of which no two are parallel. It is reqd. to draw circles to touch each of these given st. lines.

(1) Locus of centres of circles touching the st. lines AB and BC is one or other of the st. lines BI_2, I_1BI_3, the bisectors of the angles between AB and BC [Note (vi) page 188].

(2) Locus of centres of circles touching the st. lines BC, CA is one or other of the st. lines CI_3, I_2CI_1, the

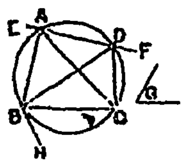
bisectors of the angles between BC and CA [Note (vi), page 188].

Let BI_3, CI_3 meet at I_3; CI_1, BI_1 at I_1, BI_2, CI_2 at I_2, and BI_2 CI_3 at I. ∴ the pts. I, I_1, I_2 and I_3 satisfy both the conditions; ∴ they are the centres of the reqd. circles. With centres I, I_1, I_2 and I_3 draw the circles as in diagram.

Thus there are four circles to touch each of the three given st. lines AB, BC, CA

Page 191.

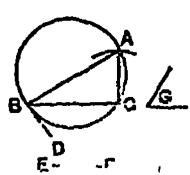
1 Let BC be the given base, G the given angle and EF the given st. line. It is reqd to describe a triangle upon BC, having its vertical angle = ∠ G and vertex on the line EF.



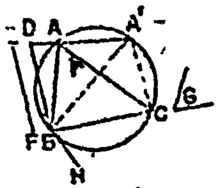
Upon BC describe a segment BADC containing an angle = the ∠G Prob. 24). Then the vertex of the reqd. triangle lies on the arc ABDC. Also the vertex lies on the st. line EF. Therefore, the pts A D where the segment BADC cuts the st. line EF represent the vertices of the reqd triangle. Join AB, AC, DB, DC. Then ABC DBC are the two reqd. triangles.

2. Let BC be the given base and G the given vertical angle. Upon BC describe a segment BAC containing an angle = the given ∠ G (Prob 24) Then the vertex of the triangle whose base is BC and the vertical angle = ∠ G lies on the arc BAC.

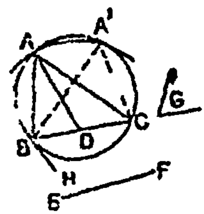
(i) Let EF denote the length of one of the sides of the triangle. With centre B and radius = EF draw an arc. Then the vertex of the reqd. triangle lies on this arc. Therefore the pt. A where this arc cuts the arc BAC is the reqd vertex. Join AB, AC Then ABC is the reqd. triangle.



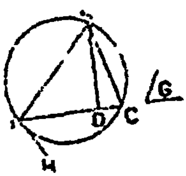
(ii) Let EF denote the length of the altitude. At B draw BD perp. to BC making BD = EF. From D draw DA′ parallel to BC. Then the vertex of the reqd. triangle also lies on the line DA′. Therefore the pts. A, A′ where AD′ cuts the arc BAC are the reqd. vertices. Join AB, AC, A′B, A′C. Then ABC, A′BC are the reqd. triangles.



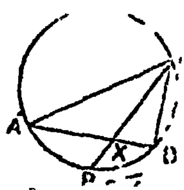
(iii) Let EF denote the length of the median which bisects BC. Bisect BC at D. With centre D and radius = EF draw an arc. Then the vertex of the reqd. triangle also lies on this arc. Therefore the pts. A, A′ where this arc cuts the arc BAC are the reqd vertices. Join AB, AC, A′B, A′C. Then ABC, A′BC are the reqd. triangles.



(iv) Let D be the foot of the perp. from the vertex to the base B C. At D draw DA perp. to BC. Then the vertex of the reqd triangle lies on the st. line DA. Therefore the pt. A where DA meets the arc BAC is the reqd. vertex. Join BA, AC. Then ABC is the reqd. triangle.

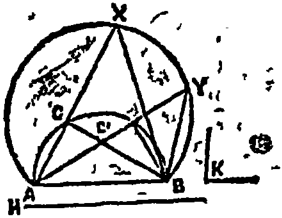


3. Construct the triangle as given in the Text Book.



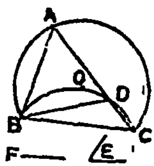
Proof—Since the arc AP = the arc PB (by construction), therefore the ∠ ACP = the ∠ PCB (Theor 43). Hence the st. line CP is the bisector of the vertical ∠ ACB which is equal to the given ∠ K. Therefore ABC is the reqd triangle.

4. Construct given in the Text Book. Join BX.



Proof—Since the $\angle$ ACB = the $\angle$ K, the $\angle$BXC $= \frac{1}{2}$ $\angle$ K, and $\angle$ A C B = $\angle$ BXC + $\angle$ XBC (Theor. 16), therefore $\angle$ XBC = $\angle$ ACB − $\angle$ BXC = $\angle$ K − $\frac{1}{2}$ $\angle$ K $= \frac{1}{2}$ $\angle$ K = the $\angle$ BXC. $\therefore$ CX = CB (Theor. 6). BC + CA = AC + CX = AX = H. Therefore ABC is the reqd. triangle. AY cuts the smaller segment at C′. Join AC′, BC′ and BY Then it can be proved that ABC′ is another such triangle.

5. Let BC be the given base, E the given angle and the line F equal to the difference of the remaining sides.



Construction—On BC describe a segment BAC containing an angle equal to E, also another segment BOC containing an angle $\angle 90° + \frac{1}{2}$ E, (Prob 24). With centre C and radius = F draw an arc cutting the arc BOC at D Join CD and produce it to meet the arc BAC at A. Join AB Then ABC is the reqd. triangle.

Proof—The $\angle$ ADB = 180° — $\angle$ BDC = 180 — (90° + $\frac{1}{2}$ E) = 90° — $\frac{1}{2}$ E

The $\angle$ B D C = $\angle$ B A D + $\angle$ ABD, (Theor. 16), therefore the $\angle$ BAD = $\angle$ BDC — $\angle$ ABD = (90° + $\frac{1}{2}$E) — E = 90° — $\frac{1}{2}$ E. Therefore the $\angle$ ADB = the $\angle$ BAD, and hence AD = AB (Theor 6). AC − AB = AC − AD = DC = F. $\therefore$ ABC is the reqd. $\triangle$.

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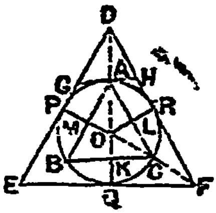
1. With any pt. O as centre and radius $=5$ cm draw a circle. At any pt. A on the circle draw a tangent GAH (Prob 22) At A make the $\angle^s$ GAB, HAC each $=60°$, the arms AB, AC meeting the circle at B C Join BC. Then ABC is the reqd. inscribed equilateral triangle.

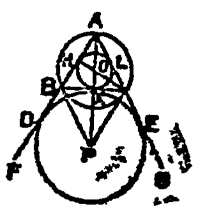


The $\angle$ GAB $=\angle$ ACB in the alt. segment (Theor. 49) $=60°$; and the $\angle$HAC$=\angle$ABC in the alt. segment (Theor. 49)$=60°$. But the $\angle^s$ GAB, BAC and CAH$=180°$ (Theor. 1). Therefore the $\angle$BAC also$=60°$. Hence the $\triangle$ ABC is equiangular, and consequently equilateral (Cor. Theor. 6).

Draw any radius OQ. At O make the $\angle^s$ QOP, QOR each$=120°$. At Q,P,R draw EF,DE,DF tangents to the circle meeting one another at D,E and F. Then DEF is the reqd. circumscribed triangle.

Since the $\angle^s$ POQ, QOR, POR$=360°$ (Cor. 3, Theor. 1) therefore the $\angle$POR$=360°-240°=120°$. Since the $\angle^s$ OPD, ORD are rt. angles (Theor 46), therefore the pts. D,P,O,R are concyclic (Converse, Theor 40): Therefore the $\angle^s$ PDR, POR$=180°$ (Theor. 40). Therefore the $\angle$PDR$=180°-120°=60°$. Similarly it can be shown that the $\angle^s$ PEQ, RFQ each$=60°$. Hence the $\triangle$DEF is equiangular, and consequently equilateral (cor. Theor. 6).

2. Draw a st. line BC$=8$ cm With centres B, C and radius$=8$ cm. draw two arcs cutting one another at A. Join AB, AC. Then ABC is the reqd. equilateral triangle.



Bisect the $\angle^s$ BAC, ACB by the st. lines AK, CH cutting one another at O, Then O is the centre of the

inscribed circle (Prob. 26). Let AK, CH meet BC, AB at K and H.

∴ AK bisects BC at rt. angles, and CH bisects AB at rt. angles (Ex. 1 page 19).

And AK and CH cut one another at O. Therefore O is also the centre of the circumscribed circle (Prob. 25)

Produce AB, AC to F and G. Bisect the $\angle^s$ CBF, BCG by the st. lines BP, CP meeting each other at P. Then P is the centre of the escribed circle (Prob. 27).

Join PK. The $\angle$ FBC $= 120° =$ $\angle$ BCG. Therefore their halves are equal, so that $\angle$ PBC $=$ $\angle$ PCB $= 60°$, hence PB $=$ PC (Theor. 6). The $\triangle^s$ BKP, PKC are identically equal (Theor. 7), because PB $=$ PC, BK $=$ KC (proved) and PK is common to both, so that the $\angle$ BKP $=$ the $\angle$ PKC and these being adjacent angles, each is a rt $\angle$. But the $\angle$ BKA is also a rt. $\angle$. Therefore AK and KP are in one st. line.

The $\triangle^s$ BAK and BKP are congruent, because $\angle$ BKA $=$ $\angle$BKP being rt. angles, $\angle$ABK $=$ $\angle$KBP being $= 60°$, and BK is common to both (Theor. 17) Therefore AK $=$ KP Since AK is the median of the $\triangle$ ABC, OK $= \frac{1}{3}$ AK, AO $= \frac{2}{3}$ AK (Cor. Proposition III, page 97).

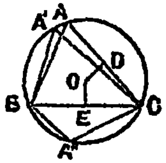
∴ AO $= 2$ OK, and KP $= 3$ OK. Hence the circum-radius OA and ex-radius PK are respectively double and treble of the in-radius OK. AK $= \sqrt{AC^2 - KC^2} = \sqrt{8^2 - 4^2} = \sqrt{48} = 6 \cdot 9$ cm. nearly.

OK $= \frac{1}{3} \times 6 \cdot 9 = 2 \cdot 3$ cm., ∴ OA $= 4 \cdot 6$ cm. and PK $=$ $6 \cdot 9$ cm.

Measure them and it will be found that OK $= 2 \cdot 3$ cm., OA $= 4 \cdot 6$ cm. and PK $= 6 \cdot 9$ cm.

(4) Draw a st. line BC $= 2 \cdot 5''$. At B, C make the $\angle^s$ CBA, BCA $= 66°$ and $50°$ respectively, the arms BA, CA meeting at A. Then ABC is the reqd. triangle. Bisect

BC, AC at E and D. At E, D draw perps. E O, D O meeting each other at O. With centre O and radius OB draw a circle which will pass through C and A also (Prob. 25). Measure O B and it will be found to be = 1·39."



(ii) Draw the $\triangle$ A′BC making $\angle$ B=72°, $\angle$ C= 44°.

(iii) Also draw the $\triangle$A″BC making $\angle$B=41°, $\angle$C= 23° but on the other side of AC. Circumscribe a circle in each case and measure the radius which will be found to be 1·39″ in each case. The vertical $\angle$ A=180° − B+C), (Theor. 16)=180°−(66°+50°)=64° in case (i).

$\angle$A′=180°—(72°+44°)= 64° in case (ii),

$\angle$A″=180°—(41°+23°)= 116° in case (iii).

Because the base BC is of the same length in all the cases, and the vertical $\angle$ A in case (i) = the vertical $\angle$ A′ in case (ii)=the supplement angle of the vertical $\angle$A″ in case (iii), therefore they lie on the same circle (Theors. 39 and 40, converses). Hence their circum-radii are equal.

4. *See figure in Ex.* 1.—With any pt. O as centre and radius=4 cm. describe a circle. Inscribe and circumscribe equilateral $\triangle^s$ ABC, DEF in and about this circle, as in Ex 1 Draw AK perp. to BC. The $\triangle^s$ ABK, AKC are congruent, because AB=AC, AO is common to both, and the $\angle$ AKB=the $\angle$ AKC being rt. angles Theor. 18), therefore BK=KC. That is, AK bisects the base BC Join OK, then it is perp. to BC (Theor. 31). Therefore AK, OK are in one st. line. Join CO and produce it to meet AB at M. It can be proved that CF is a median of the $\triangle$ ABC. OK=½ AO, (Cor Prop. III, page 97) =2 cm.

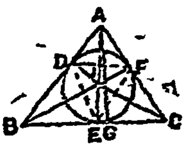
Hence KC = $\sqrt{OC^2-OK^2}=\sqrt{4^2-2^2}$ = 3·46 cm. Therefore BC=2×3·46=6·9 cm. nearly. Measure BC and it will be found to be 6·9 cm. AK=AO+OK=4 +2 = 6 cm. Hence the area of the $\triangle$ ABC=½. BC. AK=½×6·9×6=20·7 sq. cm.

Since O is the in centre of the $\triangle$DEF, DO bisects the $\angle$ EDF (Prob. 26). $\therefore$ DO when produced would bisect EF at rt. angles. (See Ex 1, page 19).

Again since both DO produced and OQ are perp. to EF from O, DO and OQ are in the same str. line; *i. e*, DOQ is a median of $\triangle$ DEF. Similarly it can be shown that FOP is also a median of $\triangle$ DEF. $\therefore$ DQ $=3$ OQ $=$ 12 cm Also FP $=3$ OP $=12$ cm. $\therefore$ FO $=\frac{2}{3}$ FP $=8$ cm. $\therefore$ QF $=\sqrt{FO^2-OQ^2}=\sqrt{8^2-4^2}=6{\cdot}9$ cm

$\therefore$ area of the $\triangle$ DEF $=\frac{1}{2}$ EF. DQ $=$ QF DQ $=12\times$ 6 9 or 82 8 sq cm. $=4\times20$ 7 sq. cm. $=4$ $\triangle$ ABC.

5. Let ABC be a triangle. Bisect $\angle^s$ABC, ACB by the st. lines BI, CI meeting at I. Then I is the centre of the inscribed circle (Prob. 26). Draw ID, IE, IF perp st on AB, BC and CA respectively.



Since ID, IE, IF are radii of the inscribed circle, therefore each of them $=r$. Join IA.

$\triangle$ IAB $=\frac{1}{2}$ ID AB $=\frac{1}{2}\,cr$, $\triangle$ IBC $=\frac{1}{2}$ IE. BC $=\frac{1}{2}$ ar, and $\triangle$ ICA $=\frac{1}{2}$ IF. AC $=\frac{1}{2}\,br$.

But the $\triangle$ ABC $=\triangle$ IBC $+\triangle$ ICA $+\triangle$ IAB

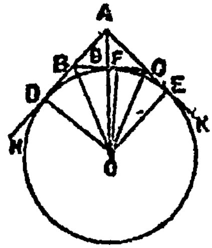
$$=\tfrac{1}{2}\,ar+br+cr=\tfrac{1}{2}\,(a+b+c)\,r.$$

In the $\triangle$ ABC, if AB $=9$ cm., BC $=8$ cm, and AC $=$ 7 cm., then ID will be found to be 2 24 cm. on measuring

$\therefore$ $\triangle$ ABC $=\frac{1}{2}\,(a+b+c)\,r=\frac{1}{2}\,(9+8+7)\times2\;24=$ 26·8 sq. cm. Draw AG perp. to BC. Then AG will be found to be 6·7 cm. (see page 111). In this case the $\triangle$ ABC $=\frac{1}{2}$ AG. BC $=\frac{1}{2}\times6{\cdot}7\times8=26$ 8 sq cm.

Thus it is evident that the formula—$\triangle$ ABC $=\frac{1}{2}\,(a+b+c)\,r$—is true.

6. Let ABC be a triangle. Produce the sides AB, AC to any pts. H and K. Bisect the $\angle^s$ CBH, BCK by the st. lines BO, CO meeting at O. Then O is the centre of the escribed circle (Prob. 27) opposite to A. From O draw OD, OE, OF perps. to AH, AK, BC respectively. Since OD, OE, OF are the radii of the escribed circle, therefore each of them $=r_1$. Join AO.



$\triangle$ ABO $=\frac{1}{2}$ OD. BA $=\frac{1}{2}cr_1$, $\triangle$ ACO $=\frac{1}{2}$ OE. AC $=\frac{1}{2}br_1$, and $\triangle$ BCO $=\frac{1}{2}$ OF BC $=\frac{1}{2}ar_1$.

But the $\triangle$ ABC $=(\triangle$ ACO $+\triangle$ ABO$)-\triangle$ BCO $=(\frac{1}{2}br_1+\frac{1}{2}cr_1)-\frac{1}{2}ar_1=\frac{1}{2}(b+c-a)r_1$

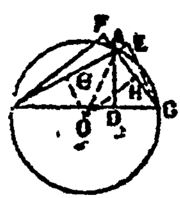
In the $\triangle$ ABC, if BC=5 cm., AC=4 cm., and AB=3 cm., then OF will be found to be 6 cm. on measurement.

$\therefore$ $\triangle$ ABC $=\frac{1}{2}(b+c-a)r_1=\frac{1}{2}(4+3-5)\times 6=6$ sq. cm.

Draw AG perp. to BC, then it will be found to be 2·4 cm. (see page 111 of the Book). In this case the $\triangle$ ABC $=\frac{1}{2}$ AG. BC $=\frac{1}{2}\times 2{\cdot}4\times 5=6$ sq. cm.

Thus it is evident that the formula $\triangle$ ABC $=\frac{1}{2}(b+c-a)r_1$, is true.

7. Construct the $\triangle$ABC in which $a=6{\cdot}3$ cm., $b=3$ cm., and $c=5{\cdot}1$ cm. (Prob. 8). Bisect the sides AB, AC at G, H respectively. At G, H draw GO, HO perps. to AB, AC meeting each other at O. Then O is the centre of the circle circumscribed about the $\triangle$ ABC (Prob. 25). Join OA and measure it; it will be found to be 3·2 cm. nearly.



From A, B, C draw AD, BF, CE perps to BC, AC, AB respectively. Measure AD, BF and CE, and it will be found that AD=2·4 cm., BF=5·04 cm, and CE=2·96 cm.

If AD, BF, CE be represented by p_1, p_2 and p_3 respectively, then $\frac{bc}{2p_1}=\frac{3\times5\cdot1}{2\times2\cdot4}=3\cdot2$ cm. nearly, $\frac{ca}{2p_2}=\frac{5\cdot1\times6\cdot3}{2\times5\cdot04}=3\cdot2$ cm. nearly and $\frac{ab}{2p_3}=\frac{6\cdot3\times3}{2\times2\cdot96}=3\cdot2$ cm. nearly.

$\therefore$ The circum-radius $OA=3\cdot2$ cm. $=\frac{bc}{2p_1}=\frac{ca}{2p_2}=\frac{ab}{2p_3}$.

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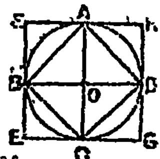


1. With centre O and radius $=1\cdot5''$ describe a circle, and in it take any two diameters AC, BD at rt. angles to each other. Join AB, BC, CD, DA. Then ABCD is the reqd. square, since its $\angle^s$ are all rt. $\angle^s$ (Theor. 41), and any side, say BC $=\sqrt{BO^2+OC^2}=\sqrt{2OB^2}=OB\sqrt{2}=1\cdot5\sqrt{2}$, or $2\cdot12''$.

Measure BC and it will be found to be $2\cdot12''$ long.

Area of the square $=BC^2=2OB^2=2\times(1\cdot5)^2$, or $4\cdot5$ sq. in.

2. *See fig in Ex* 1.—With centre O and radius $=1\cdot5''$, draw a circle, and take in it two diameters AC, BD at rt. angles to each other Draw tangents at the pts. A, B, C, D, cutting one another at E, F, G, H Then EFGH is the reqd. circumscribed square. Join AB, BC, CD, DA.

Since EH, BD and FG are at rt. angles to the same st. line AB, $\therefore$ they are parallel. Similarly EF, AC and HC are parallel.

$\therefore$ each of the figs. EFGH, EHDB, BDGF, AEFC is a Parallelogram.

$\therefore$ FG=EH=BD=AC=EF=GH.

Now $\angle$EBD is a rt. $\angle$, $\therefore$ the Parallelogram EHDB is a rectangle.

$\therefore \angle BEH$ is also a rt. $\angle$. $\therefore$ EFGH is a square.

Because the rectangles EBDH and BDGF are respectively double of the $\triangle^s$ ABD and BDC,

$\therefore$ the whole square EFGH = twice the square ABCD

3. *See fig in Ex 1.*—take a line EF = 7·5 cm., and on it describe a square (Prob. 13.). Bisect EF, FG at B, C, at B, C draw BD, CA perps. to EF, FG intersecting at O. With centre O and radius = OB describe a circle, it will touch the sides at A, B, C and D.

In the fig. BOCF since the $\angle^s$ BFC, OBC, OCF are rt. $\angle^s$, $\therefore$ BOCF is a rectangle, $\therefore$ OB = OC, also $\therefore \angle$ BOC = a rt. $\angle$. $\therefore$ each of the angles at O are rt. angles.

Fold the square about AC; then since $\angle^s$ ACF and ACG are rt. $\angle^s$, CF will fall on CG; and because CF = CG, F will fall on G. Now since $\angle^s$ CFB and CGD are equal (being rt angles), FB falls on GD

Again since $\angle^s$ BOC and COD are rt. $\angle^s$, OB falls on OD.

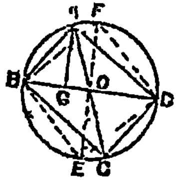
$\therefore$ B falls on D $\therefore$ OB = OD; and $\angle ODG = \angle OBF$ = a rt. $\angle$. Hence the circle touches GH at D.

Similarly, it can be proved that OA = OC (= OB, proved) and $\angle$ OAE = $\angle$ OCF = a rt. $\angle$. Hence the circle touches FG and EH at C and A. $\therefore$ it is the *inscribed* circle.

4 *See Fig. in Ex. 1.*—Draw a square ABCD on a line AB = 6 cm Join AC, BD cutting one another at O. Then the diagonals AC and BD are equal, and they bisect one another at O. $\therefore$ OA = OB = OC = OD With centre O and radius = OA describe a circle; then it will pass through B, C, D also. $\therefore$ it is the circumscribed circle.

Measure the diameter BD, and it will be found to be 8·5 cm. long.

By calculation, $BD = \sqrt{AB^2 + AD^2} = AB\sqrt{2} = 6\sqrt{2}$, or 8·48 cm.



5. With any pt. O as centre and radius $=1{\cdot}8''$ draw a circle. With any pt A as center on the circumference and radius $=3''$ draw an arc cutting the circle at D. Join AD, and at A and D draw st. lines AB, DC perp^{s}. to AD, meeting the circle at B and C. Join BC. Then ABCD is the reqd. rectangle. Join AC, BD; they are diagonals since $\angle^{s}$ ADC, BAD are rt. $\angle^{s}$ then they cut one another at centre O.

The side $DC=\sqrt{AC^2-AD^2}=\sqrt{3{\cdot}6^2-3^2}=1{\cdot}99''$ or $2''$ nearly.

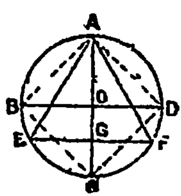
Draw the diameter FOE perp. to BD. Join FB, FD BE and ED. Then FBED is a square inscribed in the circle. Draw AG perp. to BD.

Area of the sq. FBED $=2$ the $\triangle$ FBD $=$ FO. BD; and area of the rect. ABCD $=2$ the $\triangle$ ABD $=$ AG BD.

Now, AO being the hypotenuse is greater than AG. But FO $=$ AO (being radii) $\therefore$ FO is greater than AG. $\therefore$ FO. BD is greater than AG. BD.

Therefore the area of the sq. FBED is greater than the area of the rect. ABCD.

Likewise it can be proved that the sq FBED is greater than any other inscribed rectangle.
Hence of all the rectangles inscribed in a circle the square has the greatest area.



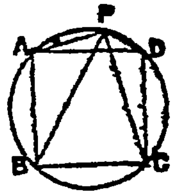
6. Let ABCD be a square and AEF an equilateral triangle inscribed in the given circle, and let a and b denote the lengths of their sides.

If r denote the radius of the given circle, then $BD^2=AB^2+AD^2$, or $(2r)^2=a^2+a^2=2a^2$, or $r^2=\frac{1}{2}a^2$.

$AE^2 = AG^2 + EG^2$ or $b^2 = (r+\frac{1}{2}r)^2 + \left(\frac{b}{2}\right)^2$, or $\frac{3}{4}b^2 = \frac{9}{4}r^2$; $\therefore r^2 = \frac{1}{3}b^2$ $\{EG = AO + OG = AO + \frac{1}{2}AO = r + \frac{1}{2}r\}$.

$\therefore \frac{1}{2}a^2 = \frac{1}{3}b^2$, $\therefore 3a^2 = 2b^2$.

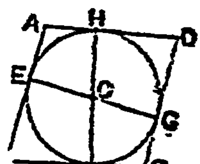
7 Let ABCD be a square inscribed in a circle and let P be any pt. on the arc AD Join PA, PD, PB, PC. It is reqd to prove that the ∠ APD = three times any one of the ∠s APB, BPC and CPD.



Proof—Since the chords AB, BC, CD are equal to one another, the arcs AB, BC, CD are also equal (Theor. 44); and hence the ∠s APB, BPC, CPD subtended by these equal arcs are equal (Theor. 43).

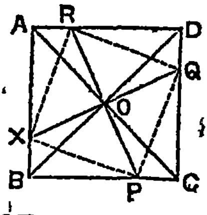
Hence the ∠ APD = ∠ APB + ∠ BPC + ∠ CPD = 3 times any one of the ∠s APB, BPC, and CPD [since these are equal angles].

8 Let O be the centre of a given circle Construction—Draw any two diameters EG and FH. At E and G draw tangents AEB and DGC. At F and H draw tangents AHD, BFC, cutting the former tangents at A, D, B, C. Then ABCD is the reqd. rhombus.



Proof—Because the ∠s AHO, OFC are rt ∠s (Theor. 46), therefore AD, BC are parallel (Theor. 13). Similarly, AB, DC are parallel. Hence the Fig. ABCD is a parallelogram, so that AD = BC and AB = DC (Theor. 21). But AD + BC = AB + DC (See Ex. 14, page 177) ∴ 2 BC = 2 DC, or BC = DC. Hence the sides AB, BC, CD, DA are all equal to one another. Hence the fig. ABCD is a rhombus.

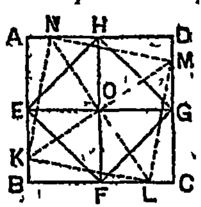
9. Let ABCD be a given square and X a point on the side AB. Cons.—Draw the diagonals AC, BD intersecting at O. Join XO and produce it to meet CD at Q. Through O draw ROP perp. to XQ meeting AD at R and BC at P. Join RQ, QP, PX and XR. Then XPQR is the reqd. square.



Proof—In the $\triangle^s$ AOR and OPC, because AO = OC (Cor. 3, Theor. 21) the $\angle$AOR = the $\angle$POC, and the $\angle$OAR = the alt. $\angle$OCP, $\therefore$ the $\triangle^s$ are congruent; so that RO = OP (Theor. 17). Similarly it can be proved that XO = OQ. Now in the $\triangle^s$ XOR and ROQ, XO = OQ, RO is common and the $\angle$XOR = $\angle$ROQ (being rt. $\angle^s$) therefore XR = RQ (Theor 4) Similarly it can be proved that QP = PX, PX = XR. Hence the fig. XPQR is a rhombus.

The $\angle$AOB = the $\angle$ROX (being rt. $\angle^s$), take away the common $\angle$AOX. $\therefore$ the $\angle$XOB = the $\angle$AOR. Now, in the $\triangle^s$ BOX and AOR, BO = AO, the $\angle XOB$ = the $\angle$AOR and the $\angle$XBO = the $\angle$RAO (each being 45°) $\therefore$ $\triangle^s$ are equal. $\therefore$ OX = OR (Theor. 17). $\therefore$ $\angle XRO$ = $\angle$RXO = 45°, since the third $\angle$ROX of the $\triangle$ ROX is a rt. $\angle$. Also OX = OP (since each = OR), $\therefore$ $\angle$OXP = $\angle$XPO = 45°. $\therefore$ $\angle$RXP = $\angle$RXO + $\angle$OXP = 90°. $\therefore$ the rhombus RXPQ is a square.

10. Let ABCD be a given square. Cons. —Bisect the sides AB, BC, CD, DA at E, F, G, H respectively. Join EF, FG, GH, HE. Then EFGH is the reqd square.



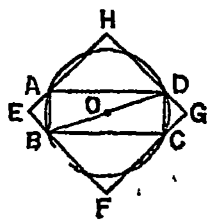
Proof—Join FH and EG. Then FH and EG are equal and intersect at rt. $\angle^s$ at O. That is the diagonals of

fig. EFGH are equal, and bisect one another at rt. $\angle^s$. $\therefore$ the fig. EFGH is a square (see proof, Ex. 9).

Let KLMN be any other inscribed square. Join the diagonals KM, NL; then these will intersect at the pt. O.

Because OK is greater than OE, and OL is greater than OF (Theor. 12), therefore KL^2, which is $=OK^2+OL^2$, is greater than EF^2, which is $=OE^2+OF^2$. That is, the sq. KLMN is less than the sq. EFGH Similarly it can be proved that any other square inscribed in the given sq. ABCD, is greater than the sq. EFGH.

Hence EFGH is the square of minimum area inscribed in the given sq. ABCD.



11. Let ABCD be a given rectangle. (i) Join BD, and describe a circle on BD as diameter. Since BAD and BCD are rt. angled triangles and BD is their common hypotenuse, therefore the circle described on BD as diameter passes through the pts. A and C (Ex. 1, page 165), and is therefore the circumscribed circle of the rectangle ABCD.

(ii) Cons.—At the pts. A and D make the $\angle^s$ DAH, ADH each $=45°$, the arms AH, DH meeting at H. Then the $\angle AHD=90°$. Through the pts. B and C draw st. lines EBF, FCG parallel to AH, DH respectively, meeting each other at F, and the st. lines HA, HD produced at the pts. E and G. Then the fig. EFGH is the reqd. square.

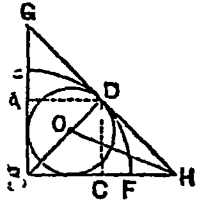
Proof—The fig. EFGH is a rectangle (by construction), therefore EH=FG, HG=EF (Theor. 21).

Now, the $\angle HAD=45°$, and the $\angle DAB=90°$, therefore the $\angle EAB=45°$. Consequently the $\angle EBA=45°$. Hence EAB is an isosceles $\triangle$. Similarly it can be shown that DCG is an isosceles $\triangle$.

Now in the $\triangle^s$ EAB and DCG, AB=DC, $\angle AEB=\angle DGC$ (being rt. $\angle^s$) and $\angle EBA=\angle GCD$ (each being 45°), therefore EA=DG (Theor. 17). But the $\angle HAD=$

the ∠HDA (by cons.), hence HA=HD (Theor. ...). Therefore HE=HG. Hence HE=HG=FG=EF. Therefore the rect EFGH is a square.

12. Let EBF be a given quadrant. (i) Bisect the ∠ EBF by the st. line BD meeting the arc EF at D. At D draw the tangent GDH to the arc EF meeting BE, BF produced at G and H respectively. Bisect the ∠BHG by the st. line HO meeting BD at O. Draw the inscribed circle of △ GBH.



Then O is the in-centre of the △ GBH (Prob. 26). Then the circle inscribed in the △GBH is the reqd circle, because it touches each of the sides BG, BH, and touches GH at D. Now, since GH is a common tangent to the circle and the arc EF at D, the circle touches the arc EF at D. Hence it is the reqd. circle.

(ii) From D draw DA, DC perps. to BG, BH respectively. Then ABCD is the reqd. square.

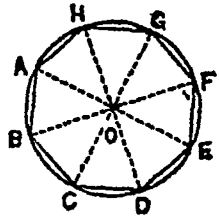
In the two △s ABD, BCD, because ∠BAD=∠BCD (being rt angles), ∠ABD=∠DBC (by cons.) and BD is common to both, ∴ the △s are identically equal (Theor. 17) ∴ AD=DC And the fig ABCD is a rectangle (by cons.) Hence it is a square, and it is inscribed in the quadrant EBF.

Page 200.

1. (i) With any pt. O as centre and radius=4 cm. describe a circle. Let COF be one of its diameters. With centre C and radius =CO draw an arc cutting the circle at B and D. Through B and D draw the diameters BOE, DOA. Join AB, BC, CD, DE, EF and FA.

Since the △s BOC, COD are equilateral, ∠DOC=60°=∠BOC. ∴ ∠AOC is also=60°=∠DOE=∠EOF=∠FOA. Thus each of the ∠s at O=60°=$\frac{1}{6}$ of 360°. ∴ ABCDEF is the reqd. regular hexagon (Prob. 30).

(*ii*) With any pt. O as centre and radius=4 cm. describe a circle. Draw any two diameters BOF, HOD at rt. angles to each other. Draw the diameters AOE, GOC bisecting the angles between the first two diameters. Then each of the angles at O is evidently $=45^\circ=\frac{1}{8}$ of 360°. Join AB, BC, CD, DE, EF, FG, GH and HA. Then ABCDEFGH is the reqd. regular octagon (Prob. 30).



(*iii*) *See fig. in Ex.* (*i*).

Bisect the angles AOB, BOC, etc. at the centre O, by OH, OK, OL, OM, ON, OG respectively. Join AH, HB, BK, KC, CL, LD, DM, ME, EN, NF, FG and GA. Then each of the angles at $O=30^\circ=\frac{1}{12}$ of 360°. $\therefore$ the fig. AHBKCLDMENFG is the reqd. regular dodecagon.

2 (*i*) With any pt. O as centre and radius$=1{\cdot}5''$ describe a circle. Inscribe a regular hexagon in this circle as in Ex. 1 (*i*); and let A, B, C, D, E, F be its angular pts. Draw tangents to the circle at these pts. meeting one another at G, H, K, L, M and N. The resulting fig. GHKLMN is the reqd. circumscribed regular hexagon.

Join OH, OK, OL, OB and OC.

Proof—Because the $\angle^s$ OBK and OCK are rt. $\angle^s$, therefore the $\angle^s$ BOC and BKC together=2 rt. $\angle^s$ (Inf. 5, Theor. 16). But the $\angle BOC=60^\circ$ [proved in Ex. 1, (*i*)], therefore the $\angle BKC=120^\circ$. Similarly it can be proved that each of the $\angle^s$ CLD, DME, ENF, FGA, AHB$=120^\circ$. Hence the fig. GHKLMN is equiangular.

Again because the circle touches the st. lines HK and KL, therefore OK bisects the $\angle HKL$ (Ex. 6, page 177).

Similarly OH, OL bisect the $\angle^s$ GHK, KLD respectively. Hence each of the $\angle^s$ OHK, OKH, OKL, OLK$=60°$.

$\therefore$ the $\triangle^s$ OHK, CKL are equiangular, and therefore equilateral. $\therefore$ HK$=$OK$=$KL.

Similarly it can be proved that KL$=$LM, and so on. Hence the fig. GHKLMN is also equilateral. Therefore GHKLMN is a regular figure.

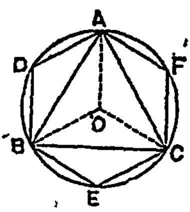
Measure all the sides of the hexagon GHKLMN and they will be found to be equal to one another; also measure the angles and it will be found that each of the angles$=120°$. Hence the fig. is regular.

(ii) With any pt. O as centre and radius$=1.5''$ describe a circle. Inscribe a regular octagon in it; and let K, L, M, N, P, Q, R, S be its angular pts. Draw tangents to the circle at these pts. cutting one another at A, B, C, D, E, F, G and H. The resulting fig. ABCDEFGH is the reqd circumscribed regular octagon.

Proof—Proceed as in the case of Ex. 2, (i).

Measure all the sides and angles of the octagon, and it will be found that all its sides are equal, and each of the angles$=135°$. Hence the fig. is regular.

3. Let O be the centre of a given circle. Inscribe a regular hexagon ADBECF in it. Join AB, BC and CA. Then ABC is the inscribed equilateral triangle in it. Let a and b denote the lengths of their sides.



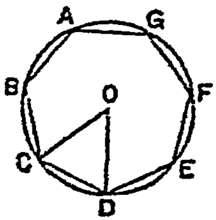
(i) Join OA, OB and OC. Then the $\angle$BOC$=2$ the $\angle$BAC (Theor. 38)$=120°$. $\therefore$ $\angle^s$ OBC and OCB are together$=180°-120°=60°$ (Theor. 16). But $\angle$OBC$=$ $\angle$OCB, since OB$=$OC, $\therefore$ each of the $\angle^s=30°$. Also

the $\angle BEC$, being the angle of a regular hexagon $=120^\circ$; and it can be proved as before that $\angle EBC = \angle ECB = 30^\circ$. Now, in the two $\triangle^s$ BOC, BEC, side BC is common, $\angle OBC = \angle EBC$, $\angle OCB = \angle ECB$ (each being $=30^\circ$). $\therefore$ two $\triangle^s$ are equal. $\therefore$ the $\triangle BOC = \frac{1}{2}$ the fig BOCE. Similarly it can be proved that $\triangle AOC = \frac{1}{2}$ the fig. AOCE, and the $\triangle AOB = \frac{1}{2}$ the fig. AOBD. Hence summing up we have the $\triangle ABC = \frac{1}{2}$ the hexagon ADBECF.

(*ii*) Because $\frac{1}{3} AB^2 = OB^2$ (Ex. 6, page 199), or $AB^2 = 3\ OB^2$, and $OB = BE$,

$\therefore AB^2 = 3\ AD^2$, that is, $a^2 = 3\ b^2$

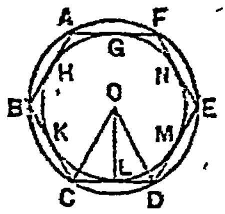
4. With any pt. O as centre and radius $= 2''$ describe a circle. At O make an $\angle COD = \frac{1}{7}$ of 360°, or $51^\circ{\cdot}4$ nearly by means of the protractor. Join CD Set off chords DE, EF, FG, GA and AB each equal to CD round the circumference. Join BC. Then ABCDEFG is the reqd. inscribed heptagon.



Because 7 times the $\angle ABC + 360^\circ = 2 \times 7$ rt. $\angle^s = 1260^\circ$ (Cor. 1, Theor. 16), therefore the $\angle ABC = \frac{1}{7}(1260^\circ - 360^\circ) = 128\frac{4}{7}^\circ$. Measure the $\angle ABC$, and a side AB, and it will be found that $\angle ABC = 128^\circ{\cdot}6$ nearly, and $AB = 1{\cdot}73''$.

Page 201.

1. Draw a st. line $CD = 2''$, and at C and D make $\angle^s$ DCB, CDE each $= 120^\circ$, making CB, DE each $= 2''$. At B and E again make the $\angle^s$ CBA, DEF each $= 120^\circ$, making BA, EF each $= 2''$. Join AF. Then ABCDEF is the reqd. regular hexagon on a side of $2''$.



Bisect the $\angle^s$ BCD, CDE by the st. lines CO, DO meeting at O. With centre O and radius OC describe a

circle; then this circle is the circumscribed circle of the hexagon ABCDEF (Prob 31)

From O draw OL perp to CD. With centre O and radius OL describe a circle; then this circle is the inscribed circle of the hexagon ABCDEF (Prob 31).

By calculation the $\angle$ OCD = 60° = $\angle$ ODC, and hence = $\angle$ COD (Theor. 16). $\therefore$ $\triangle$ OCD is equilateral. $\therefore$ OC = CD = 2″, $\therefore$ the circum-diameter = 4″. Now, CL = $\frac{1}{2}$ CD = 1″. $\therefore$ OL = $\sqrt{OC^2 - CL^2} = \sqrt{4-1} = \sqrt{3}$ = 1·73″, therefore the in-diameter = 3·46″.

Measure the circum-diameter and the in-diameter, and they will be found to be = 4″ and 3·46″ respectively.

2. *See fig. in Ex 2, (i), page* 200:—

Let O be the centre of the given circle, and let ABCDEF and GHKLMN be the inscribed and circumscribed regular hexagons. Join OH, OK, OB and OC. Let OK cut BC at P.

Then $OP = \sqrt{OC^2 - CP^2} = \sqrt{OC^2 - \frac{1}{4}OC^2} = \frac{\sqrt{3}}{2}\,OC$,

and $OC = BC = \sqrt{OK^2 - KC^2} = \sqrt{OK^2 - \frac{1}{4}OK^2} = \frac{\sqrt{3}}{2}\,OK$

$= \frac{\sqrt{3}}{2}\,HK$ Therefore $HK = \frac{2}{\sqrt{3}}\,OC$.

The $\triangle$ $OHK = \frac{1}{2}\,OB.\,HK = \frac{1}{2}\,OC \times \frac{2}{\sqrt{3}}OC = \frac{1}{\sqrt{3}}OC^2$,

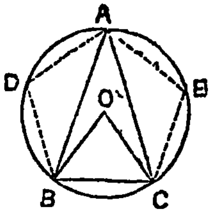
and the $\triangle$ $OBC = \frac{1}{2}\,OP.\,BC = \frac{1}{2}.\ \frac{\sqrt{3}}{2}OC \times OC = \frac{\sqrt{3}}{4}OC^2$

$= \frac{3}{4}\,\frac{1}{\sqrt{3}}\,OC^2 = \frac{3}{4}\ \triangle\ OHK$.

Now, the hexagon ABCDEF = 6 $\triangle$OBC, and the hexagon GHKLMN = 6 $\triangle$ OHK

$\therefore$ the hexagon ABCDEF = $\frac{3}{4}$ of the hexagon GHKLMN

If $OC = 10$ cm., then the area of the hexagon $ABCDEF = 6 \triangle OBC = 6 \times \frac{\sqrt{3}}{4} OC^2 = 6 \times \frac{\sqrt{3}}{4} \times (10)^2 =$ $150\sqrt{3}$ or 259 8 sq. cm.

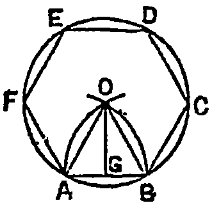


3. Let O be the centre of a given circle, and let ABC be an isosceles triangle inscribed in it, such that each of the $\angle^s$ ABC, ACB is double of the $\angle$ BAC. It is reqd. to shew that BC is a side of a regular pentagon inscribed in the circle.

The $\angle^s ABC + ACB + BAC = 180°$ (Theor. 16), or $2 \angle BAC + 2 \angle BAC + \angle BAC$, or $5 \angle BAC = 180°$. $\therefore \angle BAC = 36°$. Join OB, OC. Then the $\angle BOC =$ 2 the $\angle BAC$ (Theor. 38) $= 72° = \frac{1}{5}$ of $360°$.

Hence BC is a side of a regular pentagon inscribed in the given circle (Prob. 30).

Note—*See also Ex.* 17, *page* 171.

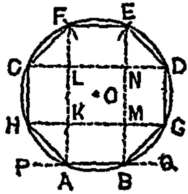


4. (*i*) Draw a st. line $AB = 4$ cm. With centres A and B, and radius $= 4$ cm. draw two arcs cutting one another at O. With centre O and radius OA or OB describe a circle. Set off chords BC, CD, DE, EF each equal to AB round the circumference of the circle. Join FA. Then ABCDEF is the reqd. hexagon [since $\triangle$ AOB is equilateral and therefore $\angle AOB = 60° = \frac{1}{6}$ of $360°$].

Area of the hexagon $ABCDEF = 6$ times the $\triangle OAB =$ $6 \times \frac{\sqrt{3}}{4} AB^2$ (proved in Ex. 2) $= 6 \times \frac{\sqrt{3}}{4} \times 16 = 41.57$ sq. cm.

(*ii*) Draw a st. line $AB = 4$ cm. Produce it both-ways to any pts. P and Q. At A and B draw AF, BE perps. to AB. Bisect the $\angle^s$ FAP, EBQ by the st. lines AH, BC respectively, making each of them $= 4$ cm.

Draw HG, CD parallel to AF or BE making each $= 4$ cm. With centres G and D, and radius $= 4$ cm. draw two arcs cutting the line AF at F and BE at E. Join GF, DE and FE Then ABCD EFGH is the reqd. octagon (since each angle $= 135°$).



Join GD cutting AF at L, BE at N. Join HC cutting AF at K and BE at M. Then the octagon is divided into 4 rt. angled isosceles triangles, four rectangles and a central square.

Now $AH^2 = AK^2 + HK^2 = 2\ AK^2$. Therefore $AK = \dfrac{AH}{\sqrt{2}} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$ cm.

$\therefore$ Area of the octagon $= 4\ \triangle AHK + 4$ rect $AB\ AK + KM^2 = 4(\frac{1}{2}\ HK.\ AK) + 4\ (AB\ \ AK) + AB^2 = 4 \times .(\frac{1}{2} \times 2\sqrt{2} \times 2\sqrt{2}) + 4 \times (4 \times 2\sqrt{2}) + 4^2 = 16 + 32\sqrt{2} + 16 = 77{\cdot}25$ sq. cm.

Page 202.

1. We know that $\pi = \dfrac{\text{circumference}}{\text{diameter}}$; $\therefore$ in case (*i*) $\pi = \dfrac{16}{5{\cdot}1} = 3{\cdot}13725$; in case (*ii*) $\pi = \dfrac{8{\cdot}8}{2{\cdot}8} = 3{\cdot}14286$; and in case (*iii*) $\pi = \dfrac{13{\cdot}5}{4{\cdot}3} = 3{\cdot}13953$ And mean of the three results $= \dfrac{3{\cdot}13725 + 3{\cdot}14286 + 3{\cdot}13953}{3} = 3{\cdot}13988$.

2. Length required for 20 complete turns $= 75{\cdot}4''$.

$\therefore$ ... ... 1 ... turn $= 3{\cdot}77''$.

Hence the circumference $= 3{\cdot}77''$. $\therefore$ $\pi = \dfrac{3{\cdot}77}{1{\cdot}2} = 3{\cdot}1417$ nearly.

3. The wheel makes 400 revolutions in 977 yards.

$\therefore$ ... ... 1 revolution ... 2·4425 yds.

Hence the circumference $= 2{\cdot}4425$ yds.

$$\therefore \pi = \frac{2{\cdot}4425 \text{ yds.}}{28 \text{ in.}} = \frac{2{\cdot}4425 \times 3 \times 12}{28} = 3{\cdot}140357.$$

Page 205.

1. The circumference of a circle $= 2\pi r$, $\therefore$ in case (i) the circumference $= 2 \times 3{\cdot}14 \times 4{\cdot}5 = 28{\cdot}3$ cm., and in case (ii) the circumference $= 2 \times 3{\cdot}1416 \times 100 = 628{\cdot}3$ cm.

2 The area of a circle $= \pi r^2$; $\therefore$ in case (i) the area $= 3\ 1416 \times (2{\cdot}3)^2 = 16\ 62$ sq. in., and in case (ii) the area $= 3{\cdot}141593 \times (10{\cdot}6)^2 = 352\ 99$ sq. in.

3. *See fig in Ex.* 1, *Page* 199.

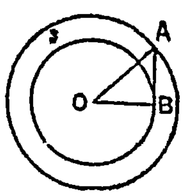
Let A B C D be the circle inscribed in the square EFGH whose side $= 3{\cdot}6''$. $\therefore$ The radius $BO = \frac{1}{2}$ $BD = \frac{1}{2}$ $EH = 1{\cdot}8$ cm.

Hence the circumference $= 2\pi r = 2 \times 3{\cdot}1416 \times 1{\cdot}8 =$ 11·31 cm.

And area $= \pi r^2 = 3{\cdot}1416 \times (1{\cdot}8)^2 = 10{\cdot}18$ sq. cm.

4. *See fig. in* 1, *Page* 199.

Since the diameter of the circle is the diagonal of the square. $\therefore$ the diagonal $= 2 \times 7 = 14$ cm And the area of the square $= \frac{1}{2}$ (product of diagonals) $= \frac{1}{2} \times 14 \times 14 = 98$ sq. cm. And the area of the circle $= \pi r^2 = \frac{22}{7} \times 7^2 = 154$ sq. cm. $\therefore$ the difference of the areas $= 154 - 98 = 56$ sq. cm.



5. Let O be the common centre of two concentric circles of radii $5{\cdot}7''$ and $4{\cdot}3''$. Then the area of the circular ring between these two circles $= \pi.\ OA^2 - \pi.\ OB^2 = \pi$ $(OA^2 - OB^2) = 3{\cdot}1416$ $(5{\cdot}7 \times 5{\cdot}7 - 4{\cdot}3 \times 4{\cdot}3)$ $= 3{\cdot}1416 \times 14 = 43{\cdot}98$ sq. in.

6. *See fig in Ex 5.*

Let O be the centre of two concentric circles, and let AB be drawn tangent to the inner circle from any point A on the outer circle. Area of a circle of radius $AB = \pi \cdot AB^2 = \pi\ (OA^2 - OB^2)$ = area of the ring (see Ex. 5).

7. *See fig. in Ex. 1, Page* 199.

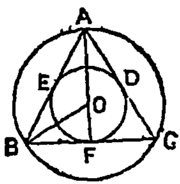
Let ABCD be the rectangle inscribed in a circle. Join AC, BD. The area of the rectangle $= AB \times AD = 8 \times 6 = 48$ sq. cm. The diameter BD of the circle $= \sqrt{AB^2 + AD^2} = \sqrt{64 + 36} = 10$ cm. ∴ The radius $= 5$ cm. Hence the area of the circle $= \pi r^2 = 3\ 1416 \times 5 \times 5 = 78{\cdot}5$ sq. cm. ∴ The area of the four segments out side the rectangle $= 78\ 5 - 48 = 30{\cdot}5$ sq. cm.

8. The area of the reqd. square = the area of the circle whose radius is $5'' = \pi \times 5^2 = 3{\cdot}1416 \times 25 = 78\ 54$ sq. in. ∴ the side of the required square $= \sqrt{78\ 54} = 8{\cdot}86'' = 8{\cdot}9''$.

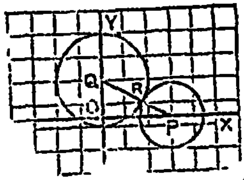
9. *See fig. in Ex 5.*

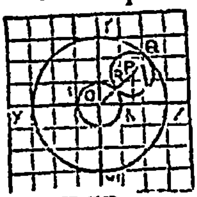
Let x'' be the radius of the smaller circle. Then the radius of the greater circle $= (x + 1)''$. ∴ The area of the ring $= \pi(x + 1)^2 - \pi X^2 = \frac{22}{7}(2x + 1) = 22$ (given). ∴ $x = 3$. ∴ the radii are $4''$ and $3''$.

10 Let ABC be the equilateral triangle whose side $= 4''$, and let ABC and EDF be the circumscribed and inscribed circles.



Then these circles are concentric, having their common centre at O. $BF = \frac{1}{2}\ BC = 2''$. ∴ $AF = \sqrt{AB^2 - BF^2} = \sqrt{16 - 4} = 2\sqrt{3}$ in. ∴ $AO = \frac{2}{3}\ AF = \frac{2}{3} \times 2\sqrt{3} = \frac{4}{3}\sqrt{3}$ in, and $OF = \frac{1}{3}\ AF = \frac{2}{3}\sqrt{3}$ in. ∴ The difference of the areas of these two circle $= \pi\ (AO^2 - OF^2) = 3{\cdot}1416 \times (\frac{16}{3} - \frac{4}{3}) = 12{\cdot}57$ sq. in.

11. Let P  and Q be the points (1·5″,0) and (0, ·8″) respectively. Join QP. Then $QP=\sqrt{OP^2+QP^2}=\sqrt{(1·5^2)+(·8)^2}=1·7''$. With centres P and Q and radii $=·7''$ and $1·0''$ draw two circles; then they will touch each other externally, because the sum of their radii $=·7''+1·0''=1·7''=$ the distance between the centres P and Q. ∴ their circumferences are $=2\times3·14\times·7=4·4''$, and $2\times3·14\times1=6·3''$ nearly. And their areas are $=3·14\times(·7)^2=1·54$ sq. in.; and $3·14\times1^2=3·14$ sq. in. nearly.

12. Let P be  the point (1·6″, 1·2″) With P as centre and radius $=1''$ describe a circle. Join OP cutting the circle at R and produce OR to meet it again at Q. From P draw PM perp to OX. Then $OP=\sqrt{PM^2+OM^2}=\sqrt{(1·6)^2+(1·2)^2}=2''$. ∴ $OR=OP-PR=2''-1''=1''$, and $OQ=OP+PQ=2''+1''=3''$. Therefore the circles described with centre O and radii 1″ and 3″ will touch the first circle, the former *externally* at R, and the latter *internally* at Q. Draw the circles as shown in the figure.

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1. *See fig in Ex* 8 page 189.

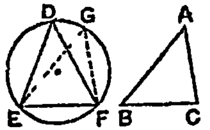
Let AB and CD be any two parallel st. lines and EF any other st. line meeting them. It is reqd. to describe circles to touch AB, CD, EF.

(i) Locus of the centres of circles touching AB and EF is one or other of the lines EO, EP which bisect the angles AEF, BEF respectively [Note VI, page 188].

(ii) Locus of the centres of circles touching CD and EF is one or other of the lines FO and FP bisecting the angles CFE and DFE respectively. [Note VI, p 188] ∴ the points O and P, where these st. lines intersect, are the

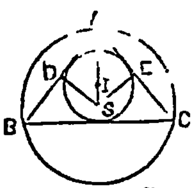
centres of the required circles (iii) Again, the centres of all circles touching two parallel straight lines lie on a line parallel to the given lines and midway between them. ∴ the points O and P are equally distant from CD, hence the radii of the two circles are equal ∴ the two circles are equal.

2. Let ABC, DEF be two triangles which have their bases BC, EF equal, and the vertical ∠BAC = the vertical ∠EDF. It is reqd. to show



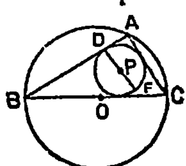
that their circumcircles are also equal. Place the △ABC over the △DEF such that the pt B falls on the pt E, and BC along EF, then because BC = EF, C will coincide with F. Let EGF represent the new position of the △ABC. Now, since the ∠EGF = the ∠EDF, the points D, G, F, E are concyclic [Converse, Theor 39]. ∴ the circum circle of the △DEF is also the circum-circle of the △EGF Therefore the circum circles of the △s DEF and ABC are equal.

3 Let ABC be a triangle, and let S and I be its circum-centre and in-centre. And let S lie on AI. It is reqd. to prove that AB = AC. Because I is the in centre, ∴ the ∠BAI = the ∠CAI (Prob 26). From S



draw SD, SE perp. to AB, AC. Then since, S is the circumcentre, ∴ D and E are the mid. pts. of AB and AC respectively (Prob. 25). In the △s SAD and SAE, the ∠SDA = the ∠SEA being rt. ∠s, the ∠SAD = the ∠SAE, and AS is common to both ∴ the △s are equal in all respects [Theor 17]. ∴ AD = AE. And since AD, AE are halves of AB, AC respectively, AB = AC.

4. Let ABC be a triangle rt. ∠d. at A, let D, d denote the diameters of the inscribed and the circumscrived circles. It is reqd. to show that $D + d = b + c$.

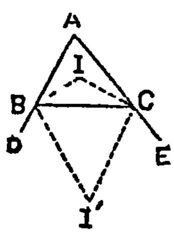


Area of the $\triangle ABC = \frac{1}{2}(a+b+c)r$; where r = radius of the inscribed circle [Ex. 5, p. 198], and it is also $= \frac{1}{2}cb$. $\therefore \frac{1}{2}cb = \frac{1}{2}(a+b+c)r$, $r = \frac{cb}{a+b+c}$. $\therefore$ $d = 2r = \frac{2cb}{a+b+c}$. Again because the $\angle$ A is a rt. angle, $\therefore a^2 = c^2+b^2$, and D = CB = a [Prob. 10]. $\therefore$ D $+ d$ $= a + \frac{2cb}{a+b+c} = \frac{ac+ab+a^2+2cb}{a+b+c} = \frac{a(c+b)+c^2+b^2+2cb}{a+b+c}$ $= \frac{a(c+b)+(c+b)^2}{a+b+c} = \frac{(c+b)(a+b+c)}{a+b+c} = c+b$.

5. *See fig in Ex 5 page* 198.

Let the inscribed circle of a $\triangle$ABC touch the sides AB, BC, CA at D, E and F respectively. It is reqd. to prove that the angles of the $\triangle$DEF are respectively $90^\circ - \frac{1}{2}$A, $90^\circ - \frac{1}{2}$B, $90^\circ - \frac{1}{2}$C. Because AF and AD are two tangents drawn from A, $\therefore$ AF = AD [Cor, Theor. 47]. $\therefore$ the $\angle$AFD = the $\angle$ADF. Now the $\angle$FAD + the $\angle$ADF + the $\angle$AFD = 180°, that is 2 $\angle$ADF + $\angle$A = 180°. $\therefore$ $\angle$ADF + $\frac{1}{2}$ A = 90° $\therefore$ the $\angle$ADF = $90^\circ - \frac{1}{2}$A. But the $\angle$ADF = the $\angle$DEF in the alt. segment [Theor. 49]. $\therefore$ the $\angle$DEF = $90^\circ - \frac{1}{2}$A Similarly, it can be proved that the $\angle$ DFE = $90^\circ - \frac{1}{2}$B, and the $\angle$FDE = $90^\circ - \frac{1}{2}$C.

6. Let ABC be a triangle and let I, I′ be the centres of the inscribed circle, and the escribed circle touching the side BC. It is reqd. to prove that I, B, I′ and C are concyclic. Because IB and IC are the internal bisectors of the $\angle$s B and C (prob. 26), and I′B, I′C are the external bisectors of the $\angle$s B and C [Prob. 27], $\therefore$ the $\angle$s IBI′ and ICI′ are rt. $\angle$s. $\therefore$ $\angle$IBI′ + $\angle$ICI′ = 2rt. $\angle$s $\therefore$ the points I, B, I′ and C are concyclic [converse, Theor. 40].

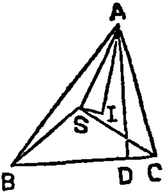


7. *See fig. in Ex. 5, page* 198.

Let ABC be a triangle, and let the inscribed circle touch the sides AB, BC. CA at D, E F, respectively.

It would be sufficient, if we prove that AC − AB = CE − BE. Because AF = AD, BE = BD, and CF = CE [Cor. Theor. 47] AC − AB = (AF + CF) − (AD + BD) = (AD + CE) − (AD + BE) = CE − BE.

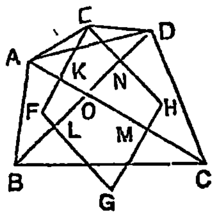
8. Let ABC be a triangle, of which the side AB is greater than AC, and let I and S be its incentre and circumcentre.



Join IS, AI and AS It is reqd. to prove that the ∠IAS = ½ (∠C − ∠B). Join SB, SC Since SB = SC (each being circum-radius), ∴ the ∠SBC = ∠SCB. Similarly ∠SBA = ∠SAB and ∠SCA = ∠SAC ∴ ∠C − ∠B = (∠ACS + ∠BCS) — ∠ABS + ∠CBS) = ∠ACS − ∠ABS [since ∠CBS = ∠BCS], = ∠CAS − ∠BAS = (∠CAI + ∠IAS) − (∠BAI − ∠IAS) = 2 the ∠IAS [∵ ∠CAI = ∠BAI]. ∴ the ∠IAS = ½ (∠C − ∠B).

(ii) From A draw AD perp. to BC Then since IA is the bisector of the ∠BAC, ∴ ∠DAI = ½ (∠C — ∠B). [Ex 3, p. 138], ∴ the ∠DAI = the ∠IAS, *i. e*, AI is the bisector of the ∠DAS.

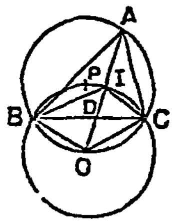
9. Let ABCD be a quadrilateral. Join diagonals AC, BD intersecting at O Bisect AO, BO CO and DO, at the pts. K, L, M N respectively. Through these pts. draw st lines EKF, FLG, GMH, HNE perps. to AO, BO, CO, DO respectively; and let them meet at the pts. E, F, G, H as in the fig Then E, F G, H are the circum-centres of the $\triangle^s$ AOD, AOB, BOC and COD respectively [Prob. 25]



It is reqd. to prove that EFGH is a parallelogram

Because EF, GH are both perps. to AC, therefore EF is parallel to GH (Ex 2, page 41). Again because EH, FG are both perps. to BD, therefore EH and FG are parallel (Ex. 2, p. 41). Hence the fig EFGH is a parallelogram.

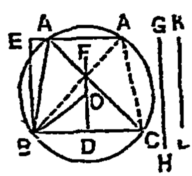
10. Let ABC be a triangle and let I be the centre of the inscribed circle. Circumscribe a circle about the $\triangle$ ABC and let P be its centre (Prob. 25). Join AI and produce it to meet the circum-circle at O. Join BI, IC.



It is reqd. to prove that O is the centre of the circle circumscribed about the $\triangle$ BIC. Join BO, CO.

Proof—Because I is the in-centre, therefore AI, BI and CI bisect the $\angle^s$ BAC, ABC and ACB respectively (Prob. 26). Therefore the $\angle$ OIC = the $\angle$ IAC + the $\angle$ ICA (Theor. 16, Obs.) = $\frac{1}{2}$ $\angle$ BAC + $\frac{1}{2}$ $\angle$ ABC. Again because the $\angle$ OCB = the $\angle$ OAB (Theor. 39) = $\frac{1}{2}$ $\angle$ BAC, and the $\angle$ BCI = $\frac{1}{2}$ $\angle$ ACB, therefore the $\angle$ OIC = the $\angle$ OCB + the $\angle$ BCI = the $\angle$ OCI, $\therefore$ OC = OI (Theor. 6). Likewise it can be proved that OB = OI. Therefore OB = OI = OC. Hence O is the centre of the circle circumscribed about the $\triangle$ BIC (Theor. 33).

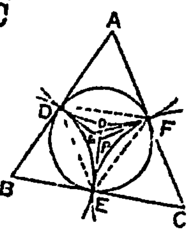
11. Let BC be the base, GH the altitude, and KL the radius of the circumscribed circle of a triangle. It is reqd. to construct the triangle.



Cons. Bisect BC at D. At D draw DF perp. to BC. Then the circum-centre lies on DF (Prob. 25). With centre C and radius = KL draw an arc cutting FD at O. With centre O and radius OC draw the circle BAC. At B draw BE perp to BC making BE = GH. From E draw EA′ parallel to BC, cutting the circle at A and A′. Join AB, AC, A′B and A′C.

Then ABC and A′BC are the two reqd. triangles satisfying the given conditions.

12. The pts. A, D, B are in one st. line; also A, F, C are in one st. line; and B, E, C are in one st. line (Theor. 48). That is, the pts D, E, F lie on the sides of the △ ABC. At D, E, draw DP, EP perps. to AB, BC, and let them meet at P. Then DP, EP are tangents at D, E Join PF. If PF is not perp to AC, let any other line FQ be perp. to AC meeting EP produced at O and DP at Q.



PD, PE are tangents to the same circle from P, ∴ PD =PE (Cor. Theor. 47). For the same reason OE=OF, and QD=QF. Now QD=QF=QO+OF=QO+OE=QO+OP+PE=QO+OP+PD=QO+OP+PQ+QD, which is absurd Hence PF is perp. to AC, and therefore tangent at F, ∴ by Cor. Theor. 47, PD=PE=PF.

∴ a circle drawn with centre P and radius=PD must pass through the pts. E and F, and also must touch the sides AB, BC, CA, at D, E, F (∵ radii PD, PE, PF= are perps. to the sides); i. e, the circle is circumscribed circle of the △ DEF, also is the inscribed circle of the △ABC.

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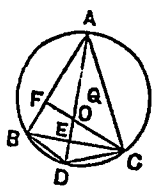
1. Let O be orthocentre of the △ABC, and let the perp. AE (from A) meet the circumcircle at D It is reqd. to prove that OE=ED.

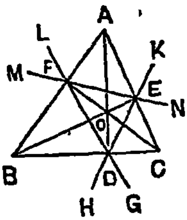


Join CO and produce it to meet AB at F. Join CD. Since ∠s AEC, AFC are rt. ∠s, ∴ ∠OCE=90°−∠EOC. and ∠OAF=90°−∠AOF, because ∠AOF=∠EOC, their complements are equal, i. e, ∠OAF=∠OCE. ∴ the ∠DCB=∠DAB (Theor. 39)=∠OCE, Now in two △s OCE, DCE,

because $\begin{cases} \angle OEC = \angle DEC \text{ (being rt. } \angle^s\text{)}, \\ \angle OCE = \angle DCE \text{ (proved)}, \\ EC \text{ is common to both}, \end{cases}$

$\therefore$ two $\triangle^s$ are identically equal ; $\therefore OE = ED$.

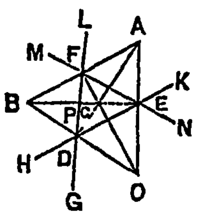
2. (i) Let **ABC** be an acute-angled triangle. Draw **AD**, **BE**, **CF**, perps from **A**, **B**, **C** on opp. sides. Join **DE**, **EF**, **FD**. Then **DEF** is the pedal $\triangle$. It is reqd. to prove that **AB**, **BC**, **CA** are *external* bisectors of the $\angle^s$ **F**, **D**, **E** of the pedal $\triangle$.



FC is the *internal* bisector of the $\angle$**DFE**. (Theor. II, page 208), and **AB** is perp to **FC**. $\therefore$ it is the *external* bisector of the same $\angle$ **DFE**, because *internal and external bisectors of an angle are at rt. angles to one another* (See Ex. 6. page 13).

Similary it can be shown that **BC**, **CA** are *external* bisectors of $\angle^s$ **FDE** and **DEF** respectively.

(ii) Let **ABC** be a $\triangle$ obtuse-angled at **C**. Draw **AE**, **BD**, **CF** perps. from **A**, **B**, **C** to opp. sides. Then **D** and **E** will be pts. on **BC** produced and **AC** produced respectively. Join **DE**, **EF**, **FD** and produce them both-ways. Then **DEF** is the pedal $\triangle$.



Now, **CF** bisects the $\angle$ **DFE** *internally* $\therefore$ **AB** (being perp to **CF**) is the *external* bisector of the $\angle$**DFE**.

Again, **AE** bisects the $\angle$ **FEK** (Theor. II on p. 208), i. e., bisects the $\angle$ **FED** *externally*, (Note at the bottom of p 208) $\therefore$ **ECB**, being perp. to **AE** bisects the $\angle$**FED** *internally*. For the same reason **DCA** being perp. to the *external* bisector **BD** of the $\angle$ **FDE**, bisects the $\angle$ **FDE** *internally*.

3. *See figs in Ex.* 2—First let us suppose the $\triangle$ **ABC** to be acute-angled as in Fig. in Ex. 2 (i). The $\angle$ **BOC** = $\angle$**FOE**. The angles **AFO**, **AEO** of the quadl.

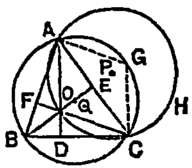
AFOE are supplementary (since each is a rt. $\angle$). $\therefore$ the fig. is concyclic (Converse, Theor. 40). $\therefore$ $\angle^s$ FOE and FAE are supplementary. But $\angle$ FOE$=\angle$BOC, $\therefore$ $\angle$ BOC and $\angle$ FAE (*i e.* the $\angle$ BAC) are supplementary.

Let the $\triangle$ be obtuse-angled at C, as in Fig. in Ex 2 (ii) Since $\angle$ ODA$=\angle$ OFA being rt. $\angle^s$, the pts. F A, O, D are concyclic (Converse, Theor 39). $\therefore$ $\angle$ FAD $=\angle$FOD in the same sagment FAOD of the circle (Theor. 39), *i.e*, the $\angle$BAC$=\angle$ BOC.

4. *See fig. in Ex* 2 (*i*)—In the $\triangle$ BOC, the lines BF, OD are perps. from vertices B, C, O to opp. sides CO, BO, BC, and they intersect at A. Hence A is the orthocentre of the $\triangle$BOC.

Similarly, it can be proved that B is the orthocentre of the $\triangle$ AOC, and C is that of the $\triangle$ AOB And O is given to be the orthocentre of $\triangle$ ABC. $\therefore$ each of the four pts O, A, B, C is the orthocentre of the $\triangle$ whose vertices are the other three

5 Let O be the orthocentre of th $\triangle$ABC. Join OA, OB, OC. Circum scribe circles about the $\triangle^s$ ABC, BOCe AOC, AOB. It is reqd to prove that all these circles are equal.



Take any pt. G on the circle circumscribing the $\triangle$ABC on the side of AC remote from B Join AG, CG

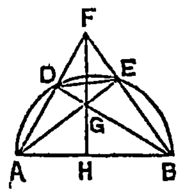
The $\angle$ BOC$=$supplement of the $\angle$ ABC [Ex. 3 (*i*), $=$the $\angle$ AGC (Theor 40) Now fold the fig. AOCG about the st. line AC, then the pt G coincides with a pt, say G′ on the same side of AC as O, and AG, CG coincides with AG′, CG′. Now $\angle$AOC$=\angle$AGC$=\angle$AG′C, $\therefore$ C and G′ lie on the same arc AOC (Converse, Theor. 39) That is, the pt G on the arc AGC coincides with a pt G′ on the arc AOC By taking other pts on the arc AGC, it can be similarly shown that each of them coincides with corresponding pts on the arc AOC. $\therefore$ the whole arc AGC coincides with the arc AOC. $\therefore$ the segment AGC$=$the segment AOC.

Similarly by taking any pt., say K, on the arc AHC and joining AK, KC, it can be proved that segment ABC =segment AHC.

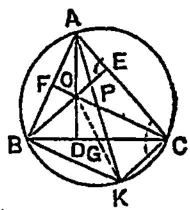
∴ by adding the circle ABCG=the circle AOCH.

In the same way it can be proved that each of the circle circumscribing $\triangle^s$ AOB, BOC is also=the circle ABCG.

6. Each of the $\angle^s$ ADB and AEB is a rt. $\angle$ (Theor. 41), *i, e*, BD and AE are perps. from B and A on opp. sides AF and BF of the $\triangle$ABF. ∴ G is the orthocentre of the $\triangle$ AFB. Now the perp. from F on AB must pass through G. ∴ FGH is perp. to AB.



7. Let ABC be a triangle. Draw BE CF perps. to AC, AB, cutting one another at O. Then O is the orthocentre. Describe a circle about the $\triangle$ ABC, and draw the diameter AK. Join BK, CK. Then BOCK shall be a parm.

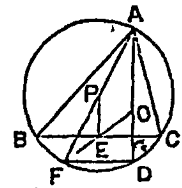


Since $\angle$ ACK is a rt. $\angle$ (Theor. 40), $\angle$A,CK=$\angle$ AEB. ∴BE *i. e.* BO and KC are parallel (Theor. 13). Similarly it can be proved that CF, *i. e.* CO and BK are parallel. ∴ fig. BOCK is a parallelogram.

8. *See Fig. in Ex.* 7—Let ABC be a $\triangle$. Draw BE, CF perps. from B, C on AC, AB, and let them cut at O. Then O is the orthocentre of $\triangle$ ABC. Describe a circle about the $\triangle$ ABC, and draw the diameter AK. Bisect BC at G. Join OG, and produce it. It is reqd. to prove that it will pass through K. Join OK.

Now BOCK is a parm. (proved in Ex. 7). ∴ its diagonals BC, OK bisect one another That is, the pt. G, the mid. pt. of BC, lies on OK. ∴ pts. O, G, K, are in same st. line; *i. e*, OG produced passes through K. Also OG=GK,

9. Let O be the orthocentre of a $\triangle$ ABC Draw AG perp. from A to BC. Then O lies on AG. Describe a circle about the $\triangle$ ABC Bisect base BC at E. Join OE, produce OE and AG to meet the circum circle at F and D. Join DF. It is reqd. to prove that DF is parallel to the base BC.

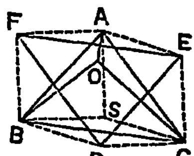


Join AF Then AF is diameter through A (*See Ex.* 8). $\therefore \angle$ ADF = 1 rt $\angle$ (Theor. 41) = $\angle$ AGB. $\therefore$ BC and FD are parallel (Theor. 13).

10. *See Fig. in Ex.* 9—Let O be the orthocentre and P the circumcentre of a $\triangle$ ABC Draw AOG perp. from A to BC Describe the circle about the $\triangle$ ABC, and draw the diameter APF Join OF cutting BC at E Then E is the mid. pt. of BC as well as of OF (proved in Ex. 8). Join PE. Then PE is perp. to BC from E (Theor 31). It is reqd. to prove that AO = 2 PE.

In the $\triangle$ AFO, P is the mid. pt. of AF, and E the mid. pt of OF. $\therefore$ PE = $\frac{1}{2}$ AO. or AO = 2 PE. (*Ex.* 3, *page* 64)

11 Let O be the orthocentre of a $\triangle$ ABC. Join OA OB, OC. Let S, D, E, F be the circumcentres of the $\triangle^s$ ABC, BOC, AOC, AOB respectively Join DE, EF, FD. It is reqd. to prove that the $\triangle$ ABC = the $\triangle$ DEF in all respects



Join SA, SB, SC, FA, FB, EA, EC, DB and DC. These are the radii of circles circumscribed about the $\triangle^s$ ABC, BOC, AOC and AOB which are all equal (proved in Ex. 5). $\therefore$ these lines are all equal to one another.

$\therefore$ each of the figs. SBDC, SAFB and SAEC is a rhombus.

$\therefore$ CD is parallel to BS, and BS is parallel to AF. $\therefore$ CD and AF are parallel, and they are also equal $\therefore$ AC = FD (Theor 20).

Similarly it can be shown that AB = ED, BC = FE. Thus we have the three sides of the $\triangle$ ABC respectively

equal to the three sides of the $\triangle$ DEF, $\therefore$ the $\triangle^s$ are equal in all respects.

12. *See Fig. in Ex.* 9.—Let A be the given vertex, O the orthocentre, and P the circumcentre. It is reqd. to construct the triangle.

The triangle is constructed if we know the base. Now from Ex. 10, we know that AO is double the perp. distance of the base from P, and is parallel to that perp. Hence we have the following construction.

Construction—Join AO, AP With P as centre and radius PA draw the circle ACDB. From P draw PE parallel to AO, making PE $= \frac{1}{2}$ AO. At E draw BEC perp to PE meeting the circle at B and C. Join AB, AC. Then ABC is the reqd. triangle.

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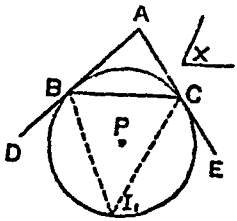
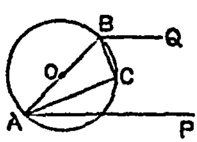


1. Let BC be the given base and X the given vertical angle; and let ABC be one of the $\triangle^s$ on the base BC, whose vertical angle A $= \angle$ X. Produce AB to any pt D and AC to any pt. E Bisect the ext. $\angle^s$ CBD and BCE by BI_1, CI_1 intersecting at I_1. Then I_1 is the excentre opp. to A. It is reqd. to find the locus of I_1.

The $\angle\ BI_1C = 90° - \frac{1}{2}$ A (*See Ex.* 7, *p* 47) = constant, since $\angle$ A is constant (being always $= \angle$ X), and BC is a given line. $\therefore$ locus of I_1 is the arc of a segment of which BC is a chord, and which contains an angle $= 90° - \frac{1}{2}$ A.



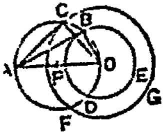
2. Let AB be the given st. line; let AP, BQ be any two parallel st. lines drawn through A and B Bisect the $\angle^s$ PAB, QBA by AC, BC, and let them meet at C. It is reqd. to find the locus of C.

The sum of the $\angle^s$ PAB and QBA $= 180°$ (Theor. 14). $\therefore$ sum of their halves $= 90°$, *i. e.*, $\angle$ABC $+ \angle$BAC $= 90°$ $\therefore$ the $\angle$ACB $= 90°$.

∴ the locus of C is the circle described on AB as a diameter (Theor. 41).

3. *See Ex* 6 *page* 165.

4. Let BDE be one of the system of concentric circles whose common centre is O. Let A be the fixed pt., and let AB be a tangent drawn from A to the circle BDE. It is reqd. to find the locus of the pt. B.



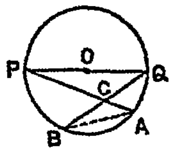
Join OB, OA. Since O and A are fixed pts., OA is a fixed st. line. And the ∠ ABO is a rt. ∠ (Theor 46). ∴ locus of B is the circle drawn on OA as diameter.

5 *See fig in Ex* 7, *page* 170. Let BCDE be the given circle, and D and E two fixed pts. on it. Let DC, EB be two such st. lines drawn from D and E, that the arc BC intercepted between them be of constant length, and let them meet at A. It is reqd. to find the locus of A.

Since arcs DE and BC are of constant lengths, the ∠s DBE and BDC subtended by these arcs at the circumference are also of constant magnitudes.

Now the ∠ DAB = the ∠ BDC − the ∠ ABD (Theor. 16). ∴ the ∠DAB or the ∠ DAE is also constant, and since the ∠ DAE stands on the fixed line DE, ∴ the locus of A is the arc of a segment of which DE is a chord, and which contains angle = the ∠BDC − the ∠ABD.

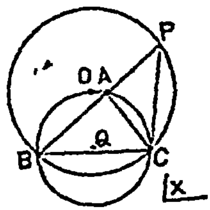
6. Let A, B be two fixed pts. on the circumference of a circle ABPQ, and let PQ be any diameter. Join AP, BQ and let them intersect at C. It is reqd. to find the locus of C.



Join AQ. Since A, B are fixed pts. arc AB is of some fixed length ∴ the ∠ AQB subtended by this arc at the circumference is of constant magnitude. And the ∠ PAQ is a rt. ∠ (Theor. 40). ∴ the ∠ ACB which

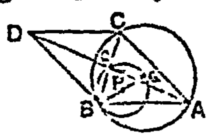
$=\angle PAQ+\angle AQB$ (Theor. 16) is also constant. And since the $\angle$ ACB stands on a fixed line AB, the locus of C is the arc of a segment of which AB is a chord, and which contains an angle $= 90^\circ +$ the $\angle$ AQB

7 Let BAC be any triangle described on the fixed base BC, and having its vertical $\angle$ BAC equal to the given $\angle$X. Let BA be produced to P such that BP=BA+AC. It is reqd to find the locus of P. Join PC.



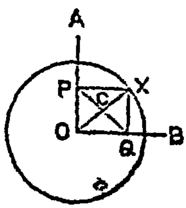
Since BP=BA+AC, therefore AP=AC, and hence the $\angle$ APC=the $\angle$ ACP (Theor. 5). The $\angle$BAC= the $\angle$ APC+ the $\angle$ ACP (Theor. 16, Obs.) = 2 the $\angle$ APC. Therefore the $\angle$ APC $= \frac{1}{2}$ the $\angle$ BAC $= \frac{1}{2}$ $\angle$ X Hence the $\angle$ APC is also constant. Therefore the locus of P is the arc of a segment on the fixed chord BC, containing an angle$=\frac{1}{2}$ the $\angle$ X

8. Let CBA be the given circle of which AB is the fixed chord. Draw any other chord AC from A, and complete the parallelogram ABDC. Draw the diagonals DA, CB cutting one another at O. It is reqd. to find the locus of O.



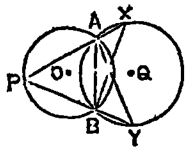
Since the diagonals of a parallelgram bisect one another, therefore O is the middle pt. of the chord BC; and since this chord passes through the fixed pt B, therefore the locus of its middle pt. O is the circle OBQ whose diameter BQ = the radius of the given circle CBA (*See Ex.* 6, *page* 165)

9 Let OA, OB be two rulers placed at rt. angles to one another, and let PQ be a position of the straight rod which slides between them. From P and Q draw PX, QX perps. to OA and OB, and let the perps. meet at X It is reqd. to find the locus of X.



The fig. POQX is by construction, a rectangle; therefore its diagonals OX, PQ are equal. Since the rod PQ is of constant length ∴ OX is also of constant length, and the pt O is a fixed pt Therefore the locus of X is the quadrant intercepted between OA and OB, of the circle whose centre is O, and whose radius=the length of the rod PQ.

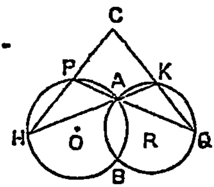
10. Let two circles intersect at A and B, and let P be any pt. on the circumference of one of them. From P two st lines PA, PB are drawn and produced to cut the other circle at X and Y Join AY, BX intersecting at C. It is reqd to find the locus of C



Because A and B are fixed pts, therefore the $\angle^s$ APB, AXB and AYB are of constant magnitudes. Therefore the ext. $\angle$ XBY, being=the $\angle$PXB+the $\angle$ XPB (Theor 16 Obs) is constant; and therefore the ext. $\angle$ ACB which is=the $\angle$CBY+the $\angle$CYB (Theor. 16, Obs.) is also constant. And this $\angle$ ACB stands on a fixed line AB.

Hence the locus of C is the arc of a segment on the fixed chord AB, containing a constant angle=$\angle$P+$\angle$X+$\angle$Y=$\angle$P+2 $\angle$X.

11. Let AHB and ABQ be any two circles intersecting at A and B. Let HAK be a fixed st. line drawn through A and terminated by the circumferences, and let PAQ be any other st. line similarly drawn. Join HP and QK, and produce them to intersect at C. It is reqd. to find the locus of C.



Since the ext $\angle$ HPQ=the $\angle$ HCK+the $\angle$ PQC (Theor. 16, Obs) therefore the $\angle$ HCK=the $\angle$ HPQ−the $\angle$ CQP. Because H, A and K are fixed pts., there-the $\angle^s$ AQK and APH which the arcs AK and AH subtend at the circumferences are of constant magnitudes.

Hence their difference is also constant. That is, the $\angle$ HCK is constant. Therefore the locus of C is the arc of a segment on the fixed chord HAK, containing an angle = the $\angle$ APH – the AQK.

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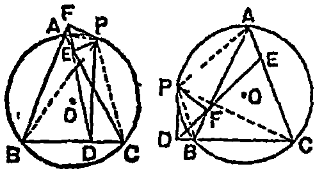


1. Let P be any pt on the circum-circle of the $\triangle$ ABC, and let PD, PF be perps drawn from P to BC and AB. Join FD. Let it cut AC at E. Join PE. It is reqd. to prove that PE is perp. to AC. Join AP, BP and CP.

Proof.—Because the $\angle$s BFP and PDB are rt. angles, therefore the pts. P, D, B, F are concyclic (Converse, Theor. 40), and hence the $\angle$ FPB = the $\angle$ FDB (Theor. 39) Also the $\angle$ ACB = the $\angle$ APB (Theor. 39). in *fig* 1, or $\angle$ ACB = 180° – $\angle$ APB in *fig*. 2.

Fig 1.—$\therefore$ the $\angle$FPA = the $\angle$FPB – the $\angle$APB = the $\angle$FDB – the $\angle$ACB = the $\angle$ DEC (Theor. 16, Obs) = the $\angle$ AEF (Theor. 3).

Fig. 2—The $\angle$ AEF = $\angle$ EDC + $\angle$ ECD = $\angle$ FPB + 180° – $\angle$ APB = 180° – ($\angle$ APB – $\angle$ FPB) = 180° – $\angle$ APF. $\therefore$ $\angle$ AEF + $\angle$ APF = 180°.

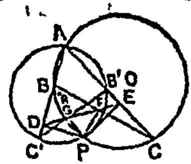
$\therefore$ the pts. A, E, P, F are concyclic. Therefore the $\angle$s AFP and AEP are supplementary (Theor. 40) in *fig* 1, or are equal (Theor. 39) in *fig* 2. But the $\angle$ AFP is a rt. angle, therefore the $\angle$ AEP is also a rt. angle. Hence PE is perp. to AC.

2. *See fig. in Ex*. 1. Let P be any such pt. that D, E, F, the feet of the perps. drawn from it on the sides of the given $\triangle$ ABC are collinear. It is reqd. to find the locus of P.

Because the $\angle$s PEA and PFA are rt. angles, therefore the pts F, A, E, P are concyclic 40), $\therefore$ the $\angle$ APF = the $\angle$ AEF in *fig*. 1, or = 180° – $\angle$ AEF in *fig*. 2, = the $\angle$ DEC.

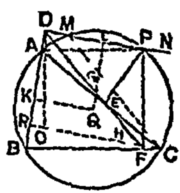
Again because the $\angle^s$ PFB, PDB are rt. angles, therefore the pts. F, B, D, P are concyclic (Converse, Theor. 40). Therefore the $\angle$ FPB = the $\angle$ FDB (Theor. 39) $\therefore$ in *fig* 1, $\angle$ FPB $-$ $\angle$ FPA $=$ $\angle$ FDB $-$ $\angle$DEC = ext $\angle$ EDB $-$ int. opp. $\angle$DEC $=$ $\angle$ECD or ACB (Theor. 16 Obs.), or, in *fig* 2, $\angle$FPB $+$ $\angle$FPA $=$ $\angle$FDB $+$ $\angle$DEC $=$ $\angle^s$EDC $+$ DEC of the $\triangle$DEC $= 180^\circ$ $-$ $\angle$ECD (or $\angle$ACB). That is, the $\angle$APB and the $\angle$ACB are equal, or supplementary. Hence, *in either case*, the pts. A, B, C and P are concyclic. Therefore the locus of P is the circum-circle of the $\triangle$ ABC.

3. Let ABC and AB′C′ be two triangles with the common $\angle$ A. Let circum-circles of these two triangles meet again at P. From P draw PD, PE, PF and PG perps. to AB, AC, BC and B′C′ respectively. It is reqd. to prove that the pts D, G, F, E are collinear.



Proof—Because PD, PF, PE are perps. drawn from P to the sides of the $\triangle$ ABC, therefore the pts. D, F and E are collinear. [Prob. V, page 212, Simson's line]. Again because PD, PG, PE are perps drawn from P on the sides of the $\triangle$ AB′C′, therefore the pts. D, G and E are collinear [Prob. V, page 212] Hence the pts. D, G, F and E are collinear.

4 Let ABC be a triangle inscribed in a given circle, and let P be any pt. on this circle. Let O be the ortho-centre of the $\triangle$ ABC. Join PO. From P draw PD, PE, PF perps to AB, AC and BC respectively. Join DF, then DF passes through E (Prob. V, page 212), let it cut OP at G. It is reqd. to prove that OP is bisected by the st. line DEF at G.

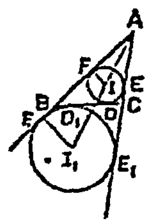


Let DP meet the circle again at M. Join MC. Produce DP to any pt. N making PN = DM. Determine Q as the circum-centre of the $\triangle$ ABC (Prob. 25), and draw QK, QL perps to AB, DN respectively. Then K and L

are the middle pts, of AB, MP respectively (Prob. 31). Join OC, then OC=2 QK (Ex. 10, page 209)=2DL=DN [because DM+ML=PN+LP, or DL=LN].

Proof—Because the $\angle^s$ AEP and ADP are rt. angles therefore the pts A, D, P and E are concyclic (Converse, Theor. 40), and hence the $\angle$PAE=the $\angle$ PDE (Theor. 39). Again because the pts A, C, P, M are concyclic, therefore the $\angle$PMC=the PAC or PAE (Theor. 39)= the $\angle$PDE. $\therefore$ DF and MC are parallel (Theor. 13). Also CHO, PMD are parallel, being perps, to the st. line A B. If CO cut DF at H, then the fig. DHCM is a parallelogram Therefore HC=DM=PN, and since OC=DN, $\therefore$ OH= DP. Also OH is parallel to DP. Therefore the fig. DOHP is a parallelogram (Theor. 20). Hence the diagonals DH, PO bisect one another at G (Cor 3, Theor. 21). Hence OP is bisected by the st. line DEF at G.

Proof of the Equalities on Prop. VI, page 213.



(i) Because from A two tangents AE, AF are drawn to the inscribed circle, therefore AE=AF (Cor. Theor. 47). Similarly it can be proved that BD=BF, and CD=CE.

Now AB+BC+CA=AF+FB+BD+DC+CE+EA =2 AE+2 BD+2 CD=2 AE+2 BC. That is $2s=2$ AE $+2a$, or 2 AE$=2s-2a$, $\therefore$ AE$=s-a=$AF. Likewise it can be proved that BD=BF$=s-b$, and CD=CE $=s-c$.

(ii) Because from A two tangents AE_1 AF_1 are drawn to the escribed circle $\therefore$ $AE_1=AF_1$ (Cor Theor. 47). Similarly it can be proved that $BF_1=BD_1$ and $CE_1=CD_1$.

$\therefore$ $AB+BC+CA=AB+BD_1+CD_1+CA=AB+BF_1+CE_1+AC_1=AF_1+AE_1=2\,AE_1$. That is $2s=2AE_1$. Therefore $AE_1=AF_1=s$.

(iii) Because $AE_1 = s$ [proved in (ii)], therefore $CD_1 = CE_1 = AE_1 - AC = s - b$.

Again because $AF_1 = s$ [proved in (ii)], therefore $BD_1 = BF_1 = AF_1 - AB = s - c$.

(iv) Because $CD = s - c$ [proved in (i), and $BD_1 = s - c$ [proved in (iii)] ; therefore $CD = BD_1$.

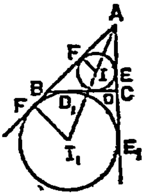
Again because $BD = s - b$ [proved in (i)], also $CD_1 = s - b$ [proved in (iii)] , therefore $BD = CD_1$.

(v) Since $AE_1 = AF_1 = s$ [proved in (ii)], and $AE = AF = s - a$ [proved in (i)] ; therefore $EE_1 = AE_1 - AE = -(s - a) = a$; and $FF_1 = AF_1 - AF = s - (s - a = a$ Hence $EE_1 = FF_1 = a$.

(vi) Area of the $\triangle ABC = \frac{1}{2}(a + b + c$ r (Ex. 5, page 198) $= rs$, since $2s = a + b + c$.

Also its area $\frac{1}{2}(b + c - a)\ r_1$ (Ex. 6, page 198) $= [\frac{1}{2}\ (a + b + c) - a]\ r_1 = s - a)\ r_1$.

(vii) If the $\angle C$ be a rt angle, then the figures IDCE and $I_1 D_1 C E_1$ would be rectangles. $\therefore$ $r = ID = CE = s - c$ [proved in (i)], and $r_1 = I_1D_1 = CE_1 = s - b$ [proved in (iii)].



Proof of the properties on Prop VII, page 214

See fig in Ex. 11, *page* 189.

(i) Because IA bisects the $\angle$ BAC (Prob. 26), and I_1A also bisects the $\angle$ BAC (prob. 27), therefore the pts A, I and I_1 are collinear. Similarly it can be proved that the pts. B, I and I_2, as well as the pts. C, I and I_3 are collinear.

(ii) Since I_1A and I_2A are the internal and external bisectors of the $\angle A$, therefore the $\angle I_1AI_2$ is a rt. angle (Ex. 6 page 13). Similarly the $\angle I_3AI_1$ is a rt. angle. Therefore the st. lines I_2A and I_3A are in one

st. line (Theor. 2). Hence the pts. I_2, A and I_3 are collinear. Similarly it can be proved that the pts. I_3, B and I_1 as well as the pts. I_1, C and I_2 are collinear.

(iii) Because AI_1 and AI_2 are the internal and external bisectors of $\angle A$, therefore I_1A is perp. to I_2A or I_2I_3. Similarly it can be proved that I_3C is perp to I_1I_2 and that I_2B is perp. to I_3I_1.

Therefore I is the orthocentre of the $\triangle I_1I_2I_3$, and ABC is the pedal triangle of the $\triangle\ I_1I_2I_3$. Therefore the $\triangle^s$ BI_1C, CI_2A, AI_3B are equiangular to one another and to the $\triangle\ I_1I_2I_3$ [Prop. 11, Cor. (ii), page 208].

(iv) If the inscribed circle touch the sides BC CA and AB at the pts. D, E and F, then the $\angle FDE = 90^\circ - \frac{A}{2}$ (Ex. 5, page 203). Also the $\angle BI_1C = 90^\circ - \frac{A}{2}$ (Ex. 7, page 47). Therefore the $\angle$ FDE = the $\angle BI_1C$. Similarly, it can be proved that the $\angle DEF$ = the $\angle AI_2C$ and that the $\angle EFD$ = the $\angle AI_3B$ Hence the $\triangle^s I_1I_2I_3$ and DEF are equiangular.

(v) Because I is the ortho-centre of the $\triangle\ I_1I_2I_3$ [proved in case (iii)], therefore of the four points I, I_1, I_2 and I_3 each is the ortho-centre of the triangle whose vertices are the other three (Ex. 4, page 209).

(vi) I is the ortho-centre of the $\triangle I_1I_2I_3$ [proved in case (iii)]; therefore the three circles which pass through two vertices of the $\triangle\ I_1I_2I_3$ and the pt. I are each equal to the circum-circle of the $\triangle I_1I_2I_3$ (Ex. 5, page 209). Hence the four circles, each of which passes through three of the pts. I, I_1, I_2, I_3 are all equal.

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1. *See fig. in Ex.* 11, *page* 189; *also in Ex.* (*i*), *page* 213. (i) It has been proved in Ex. (ii), page 213, that $AE_1 = AF_1 = s$. Similarly it can be proved

that $BD_2 = s$, also $CD_3 = s$. $BD = s - b$ [proved in Ex. (i), page 213] and $CD_1 = s - b$ [proved in Ex. (iii), page 213]

Therefore $DD_2 = BD_2 - BD = s - (s - b) = b$ and $D_1D_3 = CD_3 - CD_1 = s - (s - b) = b$. Hence $DD_2 = D_1D_3 = b$.

(ii) $CD = BD_1$ [proved in Ex. iv, page 213] and $BD_1 = s - c$ [proved in Ex. (iii) page 213] Therefore $CD = BD_1 = s - c$ Now $DD_3 = CD_3 - CD = s - (s - c) = c$, and $D_1D_2 = BD_2 - BD_1 = s - (s - c) = c$. Hence $DD_3 = D_1D_2 = c$

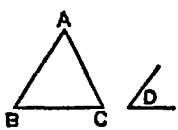
(iii) $D_2D_3 = DD_2 + DD_3 = b + c$ [from (i) and (ii)].

(iv) $DD_1 = DD_2 \sim D_1D_2 = b \sim c$.

2. *See fig in Ex* 1. I is the ortho centre of the $\triangle I_1I_2I_3$, as well as the in centre of its pedal triangle ABC And the vertices I_1, I_2 and I_3 of the $\triangle I_1I_2I_3$ are the centres of the escribed circles of the $\triangle ABC$. Therefore the orthocentre and vertices of a triangle are the centres of the inscribed and escribed circles of the pedal triangle.

3. *See fig. in Ex 1, page* 211.—Let X be the given angle and BC the given base. Let ABC be any triangle on the given base BC having the vertical $\angle A = \angle X$. Produce AB, AC to pts. D and E, and bisect the $\angle^s$ DBC, ECB by the st. lines BI_1 and CI_1 meeting at I_1. It is reqd to find the locus of I_1

Since the $\angle BI_1C = 90° - \frac{A}{2}$ (Ex. 7, p. 47), and the $\angle A$ is constant, $\therefore$ $\angle BI_1C$ is also constant. $\therefore$ the locus of I_1 is the arc of a segment on the fixed chord BC containing an angle $= 90° - \frac{A}{2}$.



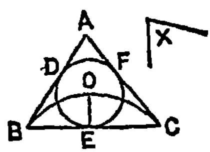
4. Let BC be the given base, and D the given angle. Let ABC be a triangle on the base BC, having its. vert. $\angle A = \angle D$. It is reqd to prove that the circum-centre of the $\triangle ABC$ is fixed.

Since the vertical ∠ BAC is constant, and the base BC is fixed, ∴ the locus of vertex A is the arc of a segment on BC as its chord, containing an angle = ∠ D (Prob. 24). But this arc circumscribes the △ ABC, ∴ the circum-circle is fixed, and hence its centre is also fixed.

5. *See fig. in Ex.* 11, *page* 189 Let ABC be a triangle on the given base BC, and having its vertical ∠ BAC = the given vertical angle. Let I_2 be the centre of the escribed circle touching the side AC It is reqd. to find the locus of I_2.

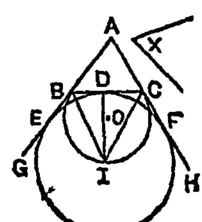
Because the ∠s I_2BI_1 and I_1AI_2 are rt. ∠s, ∴ the pts. I_1 B, A and I_2 are concyclic (Theor. 39 converse) ∴ ∠ the BI_2I_1 = the ∠ BAI_2 (Theor. 39) $= \frac{1}{2}A$ = constant. ∴ the locus of I_2 is the arc of a segment on BC as a chord containing an angle $= \frac{1}{2}A$.

6. Let BC be the given base, X the given vertical angle, and E the point of contact with the base BC of the incircle. It is reqd. to construct the triangle.



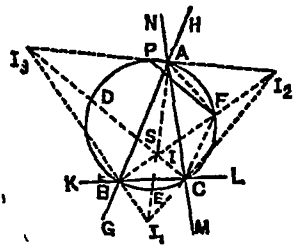
The locus of the in-centre O is the arc of a segment on BC as chord containing an angle $= 90° + \frac{1}{2}X$ (Prop. IV, p. 210). From E draw EO perp. to BC meeting this arc at O. Then O is the in-centre of the triangle and OE the in radius. With centre O and radius OE draw a circle. From B, C draw tangents to this circle (Prob. 22) and let the tangents meet at the pt. A. Then ABC is the reqd. triangle.

7. Let BC be the given base, X the given vertical angle, and D the point of contact of the escribed circle with the base BC. It is reqd. to construct the triangle.



The locus of the excentre I is the arc of a segment on BC as chord containing an angle $= 90^\circ - \frac{X}{2}$ (Ex. 1, P. 211). Draw this arc. From D draw DI perp to BC meeting this arc at I Then I is the centre and ID the radius of the escribed circle. With centre I and radius ID draw a circle, from B and C draw tangents to this circle, and produce them to meet at the pt A Then ABC is the reqd. triangle

8 Let I be the centre of the inscribed circle and I_1, I_2 and I_3 the centres of the escribed circles of the △ ABC, and let the circum-circle of the △ ABC cut II_1, II_2, II_3 at E, F and D respectively It is reqd. to show that E, F and D are the mid. pts. of II_1, II_2, II_3.



Join AF CF. The $\angle AFC = 180^\circ - B$ (Theor. 40), and the $\angle AI_2C = 90^\circ - \frac{1}{2}B$ (Ex. 7, p 47). ∴ the $\angle AFC = 2 \angle AI_2C$. Again because the $\angle^s IAI_2$ and ICI_2 are rt angles, therefore the circle on diameter II_2 passes through A and C (Ex. I, page 165). ∴ the centre of this circle lies on II_2 and since F is a pt. on II_2, such that $\angle AFC = 2 \angle AI_2C$, F must be the centre of this circle Hence II_2 is bisected at F. Similarly it can be proved that II_2 and II_3 are bisected at E and D.

9. *See fig in Ex.* 8 —Let I_2, I_3 be the centres of the escribed circles which touch the sides A C and A B of the △ A B C. It is reqd. to prove that the pts. B, C, I_2 and I_3 all lie on a circle whose centre is on the circum-circle of the △ A B C.

Proof—Because the $\angle^s I_2BI_3$ and I_2CI_3 are rt. angles, therefore the pts I_2, C, B and I_3 lie on a circle whose diameter is $I_2 I_3$ (*Ex.* 1, *page* 165). Bisect

$I_2 I_3$ at P; then P is the centre of this circle. Join FP. Because $I I_2$ is bisected at F (proved in Ex. 8), and $I_3 I_2$ at P, therefore PF is parallel to $I_3 I$, (*Ex* 2 *page* 64), hence the ext. $\angle$APF = the int. $\angle I_2 I_3$ C (Theor. 14). Again because the $\angle^s$ I A I_3 and I B I_3 are rt. angles, therefore the pts. I, A, I_3 and B are concyclic (converse, Theor. 40) $\therefore$ the A I_3 I = the $\angle$A B I (Theor 39). Therefore the $\angle$APF = the $\angle$ ABI or the $\angle$ ABF; and since they stand on the same line AF the pts. A, P, B, are concyclic (Converse, Theor. 39) But the pt. F lies on the circum-circle of the $\triangle$ ABC which passes through A and B Hence the pt. P also lies on the circum-circle of the $\triangle$ ABC.

10. *See fig in Ex 1, p.* 213.

Let A, B, C, be the three given points It is reqd. to draw with A B and C as centres, three circles which may touch one another two by two, also to show how many solutions three are.

(1) Let the inscribed circle of the $\triangle$ABC touch the sides BC CA and AB at D, E and F respectively. Then AE=AF BD=BF and CD=CE (Ex. 1, p. 213). $\therefore$ the circles described with centres A, B and C and radii AF, BD and CE respectively, will touch each other externally *two by two* (II) Let the escribed circle with I_1 as centre touch the sides AB BC and CA at the pts. F_1 D_1 and E_1 respectively. Then $AE_1=AF_1$, $BD_1=BF_1$ and $CD_1=CE_1$ (Ex. II and III, p. 213), $\therefore$ the circles described with centres A, B and C and radii AE_1 CD_1 and BF_1 will touch each other two by owo

If the escribed circle with I_2 and I_3 as centres touch the sides BC, CA, AB, at the pts. D_2 E_2, F_2 and D_3, E_3 and F_3 respectively; then it can similarly be shown that the circles described with A, B and C as centres and radii AE_2, CD_2, and BF_2, as also the circles described with centres A, B, C and radii AE_3, CD_3, BF_3 will touch each other two by two. Hence it is clear that there are *four* solutions of this problem.

11. *See fig in Ex* 1—

Let I_1, I_2 and I_3 be the centres of the three escribed circles It is reqd. to construct the triangle.

Analysis:—Let ABC be such a triangle. Join I_1I_2, I_2I_3, I_3I_1 and from I_1, I_2 and I_3 draw perps. to the opp. sides, intersecting at I_1. Since I is the orthocentre of the $\triangle I_3\ I_2\ I_3$, it is the incentre of the $\triangle ABC$; the pts. A, I, I_1, are collinear; so are the pts. B, I, I_2, and C, I, I_3, (Exs. (v) and (i), p. 214). $\therefore$ A, B, C are the feet of the perps. drawn from I_1, I_2 I_3, Hence we get the following construction.

Construction: —Join $I_1 I_2$, I_2I_3, $I_3 I_1$, from I_1, I_2, I_3 drop perps. to the opp sides, and let A, B, C, be the feet of these perps Join AB, BC, CA. Then ABC is the reqd triangle.

12 *See fig. in Ex. 1.*

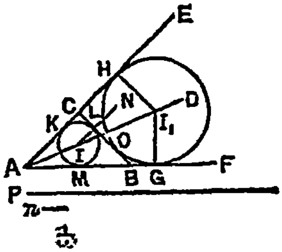
Let I be the centre of the inscribed circle, and I_3, I_2 the centres of two escribed circles. It is reqd. to construct the triangle.

Analysis —Let ABC be such a triangle. Then A, I, I_1 are collinear, so are B, I, I_2, Also if I_3 be the third ex-centre, then I_2, A, I_3, are collinear, so are I_1, B I_3; and I, C, I_3. $\therefore$ lines IC, I_1, B, I_2 A drawn from the vertices of the $\triangle II_1\ I_2$, pass through I_3, But I_3, is the otho-centre of this $\triangle$.

$\therefore$ IC, I_1 B, I_2 A are perps. drawn from the vertices of this $\triangle$ to the opp sides, and C, B, A are the feet of these perps. Hence we have the following construction.

Construction —Join II_1, I_1 I, I_2 I. From I, I_1, I_2, draw perps to the opp. sides, and let C, A, B be the feet of these perps. Join AB, BC and CA. Then ABC is the reqd. triangle.

13. Let EAF be the given vertical angle, P the semi-perimeter and r the radius of the inscribed circle. It is reqd. to construct the triangle.

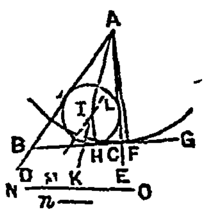


Analysis:—Let ABC be such a triangle, and let I, I_1 be the centres of the inscribed circle and the escribed circle touching the side BC. Then the points A, I and I_1, are collinear. Through I draw a st. line IL parallel to AE; then its distance from $AE=r$. From I_1 draw I_1H I_1G perps. to AE, AF, then $AH=AG=\frac{1}{2}$ P. (Ex. 11, p 313). BC is the transverse common tangent to the inscribed and escribed circles. Hence we get the following construction

Constructions:—Bisect the $\angle EAF$ by AD Draw a st line IL parallel to AE and at a distance$=r$ from it, cutting AD at I. From AE cut off AH equal to $\frac{1}{2}$ P. At H draw HI_1 perp. to AE cutting AD at I_1. With I, I_1 as centres and radii$=r$, I_1 H respectively draw two circles Then these circles will touch both AE and AF.

Draw a transverse common tangent to these two circles intersecting AE and AF at the pts. C and B respectively. Then ABC is the reqd. triangle.

14. Let DAE be the given vertical angle, NO the length of the perp. from the vertex to the base, and r the radius of the inscribed circle. It is reqd. to construct the triangle.



Analysis:—Let ABC be such a triangle, and I be the centre of its inscribed circle. Join AI, then AI bisects the $\angle$ DAE. Through I draw a st. line MIL parallel to AB; then it is at a distance $=r$ from AD.

From A draw AF perp. to BC, then AF = NO. With centres I and A and radii = r and NO respectively draw two circles. Since the former is the inscribed circle of the triangle, and since ∠ AFB is a rt. angle, BC is the direct common tangent to these two circles. Thus we get the following constructions

Constructions—Bisect the ∠DAE by the st. line AK, and draw a st. line ML parallel to AD at a distance = r from it, intersecting AK at I. With centres I and A, and radii = r and NO respectively draw two circles. Draw a direct common tangent to these two circles, and let it cut AD, AE at B, C. Then ABC is the required triangle

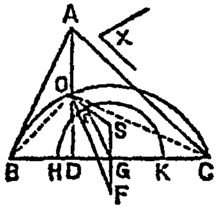
15. *See fig in Ex.* 8.

Let ABC be a triangle, and I the centre of the inscribed circle. It is required to prove that the centres of the circles circumscribed about the triangles BIC, CIA and AIB lie on the circumference of the circum-circle of the triangle ABC.

Let I_1, I_2, I_3 be the centres of the three escribed circles. Join AI_1, BI_2, CI_3, then each of them passes through I Join $I_1 I_2$, $I_1 I_3$, $I_2 I_3$; then C, A, B lie on these lines. Let the circle about the △ ABC cut II_1, II_2 II_3 at the pts. E, F, D respectively. Join AF and CF. It has already been proved in Ex. 8, that the fig. $AICI_2$ is concyclic, and that F is the centre of the circumscribing circle. Hence the centre F of the circle circumscribed about the triangle CIA lies on the circum-circle of the △ ABC Similarly it can be proved that E and D are the centres of the circles circumscribed about the △s BIC and AIB; and they lie on the circum-circle of the △ ABC.

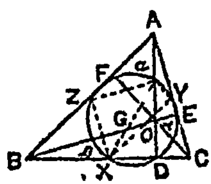
Page 218.

1. Let BC be the given base and X the given vertical angle. Suppose ABC to be a triangle on the base BC having its vertical $\angle$ A = $\angle$ X. It is reqd. to find the locus of the centre of the nine-points circle.



Since the base BC and the vertical angle is given, the circum-circle of the $\triangle$ ABC is fixed (Prob. 24). ∴ circum-radius is constant ∴ the radius of the nine-points circle = $\frac{1}{2}$ circum-radius = constant. [Property (11), page 217] And since the nine-points circle always passes through G the mid pt. of BC, its centre is always at a distance = $\frac{1}{2}$ circum-radius, from the pt. G. ∴ its locus is arc HEK of the circle whose centre is G, the mid. pt. of BC, and radius = $\frac{1}{2}$ circum-radius.

2. Let ABC be a triangle, and let O be its ortho-centre. Join AO, BO, CO. It is reqd. to prove that the nine-points circle of the $\triangle$ ABC is also the nine-points circle of each of the $\triangle^s$ AOB, BOC, COA.



The nine-points circle of the $\triangle$ ABC passes through the mid. pts. of AB, AO, BO (Theor. VIII, page 216), *i. e.*, through the three mid. pts. of the sides of the $\triangle$ AOB Since one and only one circle can pass through three points not in one st. line (Theor. 32), and since the nine-points circle of a triangle passes through the three mid. pts. of its sides, ∴ the nine-points circle of the $\triangle$ ABC must be the nine-points circle of the $\triangle$ AOB.

Similarly it can be shown that it is also the nine-points circle of each of the $\triangle^s$ BOC and COA.

3. *See fig in Ex.* II, *page* 189 Let I, I_1, I_2, I_3 be the centres of the inscribed and the escribed circles of a $\triangle$ ABC.

It is reqd to prove that the circle circumscribed about the $\triangle$ ABC is the nine-points circle of each of the $\triangle^s$ $II_1 I_2$, $II_2 I_3$, $I I_3 I_1$ and $I_1 I_2 I_3$.

∴ From Theor. VIII on page 216 and Theor. 32 we know that in a triangle the circle passing through the feet of the perps. drawn from its vertices to the opp. sides, is the nine-points circle of the $\triangle$.

It can be easily seen that in each of the $\triangle^s$ $I I_1 I_2$, $II_2 I_3$, $II_3 I_1$, $I_1 I_2 I_3$, A, B, C are the feet of the perps. drawn from the vertices to the opp. sides. Hence the circle through A, B, C, is the nine-points circle of each of the above triangles.

4. It is reqd to prove that all triangles which have the same orthocentre and the same circumscribed circle, have also the same nine-points circle.

Since all the $\triangle^s$ have the same circum-circle, their common circum-centre is a fixed pt, and common circum-radius is of constant length. Also their common ortho centre is a fixed pt.

∴ the centre of the nine-points circle, which is the mid. pt. of the st line joining the orthocentre and the circum-centre, is a fixed pt. also. And the radius of the nine-points circle = half the common circum-radius = constant.

Hence, all the triangles have the same nine-points circle.

5. *See fig in Ex. 2 (i), page* 209. Let ABC be a triangle having its base, BC = the given base and the $\angle$ ABC = the given vertical angle. Let DEF be its pedal triangle. It is reqd. to prove that one angle and one side of the pedal $\triangle$ are constant.

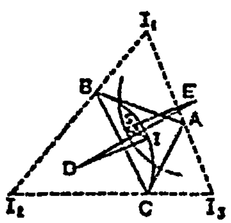
Join AD, BE, CF intersecting at O. Since FO bisects the $\angle$ EFD, and EO bisects the $\angle$ FED, ∴ the $\angle$ FOE = $90^\circ + \frac{1}{2}$ the $\angle$ FDE. (Ex. 6, page 47).

In the quadl. AF OE, since $\angle$ AFO + $\angle$ AEO = $90^\circ + 90^\circ = 180^\circ$, $\therefore$ the other two $\angle^s$ FAE + FOE $= 180^\circ$, or $\angle$ FOE $= 180^\circ - \angle$ FAE $= 180^\circ - \angle$ A $\therefore 90^\circ + \frac{1}{2}$ the $\angle$ FDE $= 180^\circ - \angle$ A. $\therefore \frac{1}{2}$ the $\angle$ FDE $= 90^\circ - \angle$ A = constant, since $\angle$ A is given to be constant. $\therefore$ $\angle$FDE is also constant.

Again because the base and the vertical angle of the $\triangle$ ABC are given, $\therefore$ its circum-circle is fixed (Prob, 24). $\therefore$ its circum-radius is of constant length. $\therefore$ radius of the nine-points circle, which is = half the circum-radius, is also of constant length ; *i. e.*, the nine-points circles of all the $\triangle^s$ whose base = the given base, and vertical $\angle$ = the given vertical $\angle$, are all equal to one another.

Now EF is a chord of the nine-points circle, and it subtends a constant angle FDE at circumferrence. $\therefore$ it is of constant length (Theors. 42 and 45).

Thus one angle FDE, and one side EF of the pedal $\triangle$ DEF are constant



6. Let ABC be a triangle on the given base BC, and having its vertical angle BAC = the given vertical angle. Let N, I, I_1, I_2, I_3 be its circum-centre, in-centre, and ex-centres respectively. It is reqd. to find the locus of the circum-centre of the $\triangle I_1 I_2 I_3$. Join $I_1 I_2$, $I_2 I_3$, $I_2 I_1$. Then A, B C lie on these lines. Also I is is the ortho-centre of the $\triangle I_1 I_2 I_3$ [Property (v), page 214]. And since A, B, C are the feet of the perps. drawn from the vertices I_2, I_3 I_1 on opp. sides ; $\therefore$ the circle through A, B, C, *i. e* the circum-circle of the $\triangle$ABC is the nine-points circle of the $\triangle I_1 I_2 I_3$. $\therefore$ N is the centre of the nine-points circle of the $\triangle I_1 I_2 I_3$. Join IN and produce it to S making NS = IN. Then S is the circumcentre of the $\triangle I_1 I_2 I_3$ [Property (i), page 217].